Low regularity exponential-type integrators for the "good" Boussinesq equation with rough solutions

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Outline

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- For numerical algorithms, it requires regularity assumptions of the exact solutions to attain the desired convergence, e.g.,

$$u_t = \frac{u^{j+1} - u^j}{t_{j+1} - t_j} + O(\tau^2)$$

is valid when u_{tt} is bounded.

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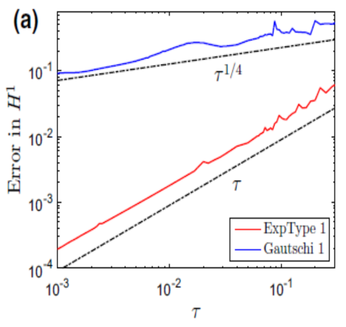

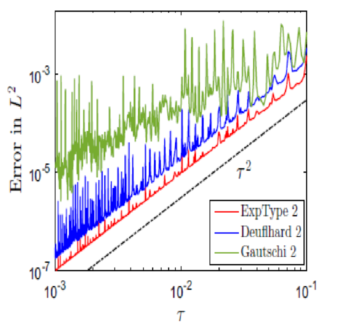

Figure 1: Convergence of the methods for rough solutions.

Low regularity integrators

Convergence

Suppose the real solution $u \in H^{r+s}$, the numerical solution u^n satisfies

$$\|u(t_n) - u^n\|_{H^r} \lesssim \tau^{\alpha}, \quad \alpha > 0.$$

α–convergence rate; s–order of additional derivatives required.

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- Dirac equation, Navier-Stokes equation, KGZ system ...

The “good” Boussinesq (GB) equation

We consider the GB equation with periodic boundary conditions

$$\begin{cases} z_{tt} + z_{xxxx} - z_{xx} - (z^2)_{xx} = 0, \quad x \in \mathbb{T}, \quad t > 0, \\ z(0,x) = \phi(x), \quad z_t(0,x) = \psi(x), \end{cases}$$

where the torus $\mathbb{T} = [-\pi, \pi]$, $\phi(x)$ and $\psi(x)$ are given initial data.

- propagation of dispersive shallow water waves;

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- propagation of dispersive shallow water waves;
- small oscillations of nonlinear beams
- two-way propagation of water waves in a channel

Classical exponential integrator

$$z_{tt} + z_{xxxx} - z_{xx} - (z^2)_{xx} = 0.$$

- Duhamel's formula:

$$\begin{aligned} z(t_n+\tau) &= L(z(t_n), z_t(t_n)) + \frac{\partial_x^2}{\sqrt{\partial_x^4 - \partial_x^2}} \int_0^\tau \sin\left((\tau - s)\sqrt{\partial_x^4 - \partial_x^2}\right) z^2(t_n+s) ds \\ &\approx L(z(t_n), z_t(t_n)) + \frac{\partial_x^2}{\sqrt{\partial_x^4 - \partial_x^2}} \int_0^\tau \sin\left(\sqrt{\partial_x^4 - \partial_x^2}(\tau - s)\right) z^2(t_n) ds \\ &= L(z(t_n), z_t(t_n)) + \partial_x^2 \left(1 - \cos\left(\tau\sqrt{\partial_x^4 - \partial_x^2}\right)\right) z^2(t_n), \end{aligned}$$

where $L(f,g) = \cos\left(\tau\sqrt{\partial_x^4 - \partial_x^2}\right) f + \frac{\sin\left(\tau\sqrt{\partial_x^4 - \partial_x^2}\right)}{\sqrt{\partial_x^4 - \partial_x^2}} g.$

Noticing

$$z(t_n+s) - z(t_n) \sim s\partial_t z(t_n) \sim s\partial_{xx} z(t_n),$$

hence two additional derivatives (in space) are required to promise the first-order convergence. Similarly four additional derivatives are needed to obtain the second-order convergence.

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- Aim of this talk: decrease the order of additional derivatives requirement as much as possible.

Preprocessing of the GB equation

- Homogenization for the GB equation (zero-mode): $\mathcal{F}_0(z_{tt}) = \partial_{tt}\mathcal{F}_0(z) = 0 \Longrightarrow \mathcal{F}_0(z) = at + b$, where

$$a = \mathcal{F}_0(z_t(0,\cdot)) = \frac{1}{2\pi}\int_{\mathbb{T}} \psi(x)dx, \quad b = \mathcal{F}_0(z(0,\cdot)) = \frac{1}{2\pi}\int_{\mathbb{T}} \phi(x)dx.$$

Plugging $z = \mathcal{F}_0(z) + \check{z}$ into GB, denote $\langle\partial_x^2\rangle := \sqrt{\partial_x^4 - \partial_x^2} \Longrightarrow$

$$\begin{cases} \check{z}_{tt} + \langle\partial_x^2\rangle^2 \check{z} - (2at+2b)\check{z}_{xx} - (\check{z}^2)_{xx} = 0, \quad x \in \mathbb{T}, \quad t > 0, \\ \check{z}(0,x) = \phi(x) - b, \quad \check{z}_t(0,x) = \psi(x) - a. \end{cases}$$

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- Equivalent first-order system: setting $u = \check{z} - i\langle \partial_x^2 \rangle^{-1}\check{z}_t$, $v = \overline{\check{z}} - i\langle \partial_x^2 \rangle^{-1}\overline{\check{z}}_t$,

$$\begin{cases} i\partial_t u = -\langle \partial_x^2 \rangle u + B\left[\frac{1}{4}(u + \bar{v})^2 + (at + b)(u + \bar{v})\right], \\ i\partial_t v = -\langle \partial_x^2 \rangle v + B\left[\frac{1}{4}(\bar{u} + v)^2 + (at + b)(\bar{u} + v)\right], \end{cases}$$

where $B := \langle \partial_x^2 \rangle^{-1}\partial_x^2$. Noticing $z \in \mathbb{R} \Longrightarrow u = v$, thus we get

$$i\partial_t u = -\langle \partial_x^2 \rangle u + B\left[\frac{1}{4}(u + \bar{u})^2 + (at + b)(u + \bar{u})\right],$$

and $\check{z} = \frac{1}{2}(u + \overline{u})$, $\check{z}_t = \frac{i}{2}\langle \partial_x^2 \rangle(u - \overline{u})$.

Introduction of twisted variable

- Twisted variable: $\langle \partial_x^2 \rangle \sim -\partial_x^2$, $w(t) = \mathrm{e}^{it\partial_x^2} u(t)$, then

$$\partial_t w = iAw - \frac{i}{4}\mathrm{e}^{it\partial_x^2} B(\mathrm{e}^{-it\partial_x^2} w + \mathrm{e}^{it\partial_x^2} \overline{w})^2 - i(at+b)\mathrm{e}^{it\partial_x^2} B(\mathrm{e}^{-it\partial_x^2} w + \mathrm{e}^{it\partial_x^2} \overline{w}),$$

where $A = \langle \partial_x^2 \rangle + \partial_x^2, \quad B = \langle \partial_x^2 \rangle^{-1} \partial_x^2$ are bounded. Hence

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- Duhamel's formula:

$$\begin{aligned}
w(t_n+\tau) &= \mathrm{e}^{i\tau A}w(t_n) \\
&\quad - \frac{i}{4}B\int_0^\tau \mathrm{e}^{i(\tau-s)A}\mathrm{e}^{i(t_n+s)\partial_x^2}(\mathrm{e}^{-i(t_n+s)\partial_x^2}w(t_n+s) + \mathrm{e}^{i(t_n+s)\partial_x^2}\overline{w}(t_n+s))^2 ds \\
&\quad - iB\int_0^\tau \mathrm{e}^{i(\tau-s)A}\mathrm{e}^{i(t_n+s)\partial_x^2}[a(t_n+s)+b](\mathrm{e}^{-i(t_n+s)\partial_x^2}w(t_n+s) + \mathrm{e}^{i(t_n+s)\partial_x^2}\overline{w}(t_n+s))ds \\
&= \mathrm{e}^{i\tau A}w(t_n) - \frac{i}{4}B\int_0^\tau \mathrm{e}^{i(t_n+s)\partial_x^2}(\mathrm{e}^{-i(t_n+s)\partial_x^2}w(t_n) + \mathrm{e}^{i(t_n+s)\partial_x^2}\overline{w}(t_n))^2 ds \\
&\quad - iB\int_0^\tau \mathrm{e}^{i(t_n+s)\partial_x^2}[a(t_n+s)+b](\mathrm{e}^{-i(t_n+s)\partial_x^2}w(t_n) + \mathrm{e}^{i(t_n+s)\partial_x^2}\overline{w}(t_n))ds + \mathcal{R}_0(\tau^2),
\end{aligned}$$

where we used $\|(\mathrm{e}^{itA}-1)f\|_r \le |t|\|f\|_r$.

Integral of the nonlinear interactions

- The approximation of $u(t_n+\tau)$:

$$u(t_n+\tau) = \mathrm{e}^{i\tau\langle\partial_x^2\rangle}u(t_n) - \frac{i}{4}B^\tau\left[L_0^\tau(u(t_n)) + L_1^\tau(u(t_n)) + 2L_2^\tau(u(t_n))\right] - i\tau(at_n+b)B^\tau\big(u(t_n) + \psi_1(2i\tau\partial_x^2)\overline{u}(t_n)\big) + \mathcal{R}_0(\tau^2),$$

where $\psi_1(y)=\int_0^1 e^{ys}ds$, and the operators are defined by

$$B^\tau(f) = B\mathrm{e}^{-i\tau\partial_x^2}f = \langle\partial_x^2\rangle^{-1}\partial_x^2\mathrm{e}^{-i\tau\partial_x^2}f, \quad L_0^\tau(f) = \int_0^\tau \mathrm{e}^{is\partial_x^2}\big(\mathrm{e}^{is\partial_x^2}\overline{f}\big)^2 ds,$$

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- Exact integration for L_1 and L_2 for f with $\widehat{f}_0 = 0$:

$$L_1^\tau(f) = \sum_k \sum_{k_1+k_2=k} \int_0^\tau \mathrm{e}^{-is(k^2-k_1^2-k_2^2)}ds\,\widehat{f}_{k_1}\widehat{f}_{k_2}\mathrm{e}^{ikx}$$
$$= \Big(\sum_k \sum_{\substack{k_1+k_2=k\\ k_1\neq 0, k_2\neq 0}} \frac{\mathrm{e}^{-2i\tau k_1k_2}-1}{-2ik_1k_2} + \sum_k \sum_{\substack{k_1+k_2=k\\ k_1=0 \text{ or } k_2=0}} \int_0^\tau ds\Big)\widehat{f}_{k_1}\widehat{f}_{k_2}\mathrm{e}^{ikx}$$
$$= \sum_k \sum_{\substack{k_1+k_2=k\\ k_1\neq 0, k_2\neq 0}} \frac{\mathrm{e}^{-2i\tau k_1k_2}-1}{-2ik_1k_2}\widehat{f}_{k_1}\widehat{f}_{k_2}\mathrm{e}^{ikx} = \frac{i}{2}\left[(\partial_x^{-1}f)^2 - \mathrm{e}^{i\tau\partial_x^2}(\mathrm{e}^{-i\tau\partial_x^2}\partial_x^{-1}f)^2\right].$$

First-order schemes

- Calculation of the term $L_0^\tau(f)$:

$$L_0^\tau(f) = \sum_{k} \sum_{k_1+k_2=k} \int_0^\tau \mathrm{e}^{-is\Phi} ds \widehat{\overline{f}}_{k_1} \widehat{\overline{f}}_{k_2} \mathrm{e}^{ikx},$$

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- First-order scheme: $u^{n+1} = \Psi_1^\tau(u^n)$

$$\Psi_1^\tau(f) = \mathrm{e}^{i\tau\langle\partial_x^2\rangle} f - \frac{i}{4} B^\tau \left[I_0^\tau(f) + L_1^\tau(f) + 2L_2^\tau(f) \right] \\ - i\tau(at_n + b) B^\tau \left(f + \psi_1(2i\tau\partial_x^2)\overline{f} \right),$$

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- Reconstruction of z:

$$z^n = \frac{1}{2}(u^n + \overline{u^n}) + at_n + b, \quad z_t^n = \frac{i}{2}\langle\partial_x^2\rangle(u^n - \overline{u^n}) + a.$$

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First-order scheme I (2019 A. Ostermann & C. Su NM)

- Approximate the term $L_0^\tau(f)$: for $\Phi = k^2 + k_1^2 + k_2^2 = 2k^2 - 2k_1k_2$,

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&= \sum_k \sum_{k_1+k_2=k} \int_0^\tau \left(\mathrm{e}^{-2isk^2} + \mathrm{e}^{-2isk^2} (\mathrm{e}^{2isk_1k_2} - 1) \right) ds \widehat{\overline{f}}_{k_1} \widehat{\overline{f}}_{k_2} \mathrm{e}^{ikx} \\
&= \tau \psi_1(2i\tau\partial_x^2)(\overline{f})^2 + P_1^\tau(f).
\end{aligned}
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L_0^\tau(f) &= \sum_k \sum_{k_1+k_2=k} \int_0^\tau \mathrm{e}^{-is\Phi} ds \widehat{\overline{f}}_{k_1} \widehat{\overline{f}}_{k_2} \mathrm{e}^{ikx} \\
&= \sum_k \sum_{k_1+k_2=k} \int_0^\tau \left(\mathrm{e}^{-2isk^2} + \mathrm{e}^{-2isk^2} (\mathrm{e}^{2isk_1k_2} - 1) \right) ds \widehat{\overline{f}}_{k_1} \widehat{\overline{f}}_{k_2} \mathrm{e}^{ikx} \\
&= \tau \psi_1(2i\tau\partial_x^2)(\overline{f})^2 + P_1^\tau(f).
\end{aligned}
$$

First-order scheme I (2019 A. Ostermann & C. Su NM)

- Approximate the term $L_0^\tau(f)$: for $\Phi = k^2 + k_1^2 + k_2^2 = 2k^2 - 2k_1k_2$,

$$\begin{aligned} L_0^\tau(f) &= \sum_k \sum_{k_1+k_2=k} \int_0^\tau \mathrm{e}^{-is\Phi} ds \widehat{\overline{f}}_{k_1} \widehat{\overline{f}}_{k_2} \mathrm{e}^{ikx} \\ &= \sum_k \sum_{k_1+k_2=k} \int_0^\tau \left(\mathrm{e}^{-2isk^2} + \mathrm{e}^{-2isk^2} (\mathrm{e}^{2isk_1k_2} - 1) \right) ds \widehat{\overline{f}}_{k_1} \widehat{\overline{f}}_{k_2} \mathrm{e}^{ikx} \\ &= \tau \psi_1(2i\tau\partial_x^2)(\overline{f})^2 + P_1^\tau(f). \end{aligned}$$

- Estimate of the remainder: by $|\mathrm{e}^{ix} - 1| \le |x|$, $x \in \mathbb{R}$, $\Longrightarrow$

$$\begin{aligned} \|P_1^\tau(f)\|_r &= \Big(\sum_{l\in\mathbb{Z}} (1+l^2)^r \Big| \sum_{k_1+k_2=l} \widehat{\overline{f}}_{k_1} \widehat{\overline{f}}_{k_2} \int_0^\tau \mathrm{e}^{-2isk^2} (\mathrm{e}^{2isk_1k_2} - 1) ds \Big|^2 \Big)^{1/2} \\ &\lesssim \tau^2 \Big(\sum_{l\in\mathbb{Z}} (1+l^2)^r \Big(\sum_{k_1+k_2=l} |k_1||k_2||\widehat{\overline{f}}_{k_1}||\widehat{\overline{f}}_{k_2}| \Big)^2 \Big)^{1/2} \\ &\lesssim \tau^2 \|f\|_{r+1}^2, \quad r > 1/2. \end{aligned}$$

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&\lesssim \tau^2 \|f\|_{r+1}^2, \quad r > 1/2.
\end{aligned}
$$

- Convergence of Scheme I obtained by taking $I_0^\tau = \tau\psi_1(2i\tau\partial_x^2)(\overline{f})^2$:

$$
\|u^n - u(t_n)\|_r \lesssim \tau, \quad \text{for} \quad r > 1/2, \quad u \in H^{r+1}.
$$

First-order scheme II (2023 H. Li & C. Su IMA)

- Key relation of the Fourier coefficients (2022 Y. Wu & F. Yao MC):

$$\frac{k_1 + k_2}{k} = 1, \quad \text{and} \quad \Phi = 2k_2^2 + 2kk_1 = 2k_1^2 + 2kk_2.$$

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- Calculation of $L_0^\tau(f)$ with $\widehat{f}_0=0$:

$$L_0^\tau(f)-\mathcal{F}_0(L_0^\tau(f))=\sum_{k\neq 0}\sum_{k_1+k_2=k}\int_0^\tau \mathrm{e}^{-is(k^2+k_1^2+k_2^2)}ds\widehat{\overline{f}}_{k_1}\widehat{\overline{f}}_{k_2}\mathrm{e}^{ikx}$$

$$=\sum_{k\neq 0}\sum_{k_1+k_2=k}\frac{k_1}{k}\int_0^\tau \mathrm{e}^{-is\Phi}ds\widehat{\overline{f}}_{k_1}\widehat{\overline{f}}_{k_2}\mathrm{e}^{ikx}+\sum_{k\neq 0}\sum_{k_1+k_2=k}\frac{k_2}{k}\int_0^\tau \mathrm{e}^{-is\Phi}ds\widehat{\overline{f}}_{k_1}\widehat{\overline{f}}_{k_2}\mathrm{e}^{ikx}$$

$$=T_1^\tau(f)+T_2^\tau(f).$$

By symmetry, we have $T_1^\tau(f)=T_2^\tau(f)$.

Derivation

- Decomposition:

$$e^{-is\Phi} = e^{-2is(k_2^2+kk_1)} = e^{-2isk_2^2} + e^{-2iskk_1} - 1 + (e^{-2isk_2^2} - 1)(e^{-2iskk_1} - 1).$$

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- $T_1^\tau(f)$ can be written as

$$T_1^\tau(f) = \sum_{k\neq 0} \sum_{k_1+k_2=k} \frac{k_1}{k} \int_0^\tau e^{-2is(k_2^2+kk_1)} ds \widehat{\overline{f}}_{k_1} \widehat{\overline{f}}_{k_2} e^{ikx}$$

$$= \sum_{k\neq 0} \sum_{\substack{k_1+k_2=k \\ k_2\neq 0}} \frac{k_1}{k} \int_0^\tau e^{-2isk_2^2} ds \widehat{\overline{f}}_{k_1} \widehat{\overline{f}}_{k_2} e^{ikx} + \sum_{k\neq 0} \sum_{\substack{k_1+k_2=k \\ k_2\neq 0}} \frac{k_1}{k} \int_0^\tau (e^{-2iskk_1} - 1) ds \widehat{\overline{f}}_{k_1} \widehat{\overline{f}}_{k_2} e^{ikx}$$

$$+ \sum_{k\neq 0} \sum_{k_1+k_2=k} \frac{k_1}{k} \int_0^\tau \left(e^{-2isk_2^2} - 1\right) \left(e^{-2iskk_1} - 1\right) ds \widehat{\overline{f}}_{k_1} \widehat{\overline{f}}_{k_2} e^{ikx}$$

$$:= F_1^\tau(f) + F_2^\tau(f) + P_2^\tau(f),$$

where F_1 and F_2 can be integrated exactly

$$F_1^\tau(f) = -\frac{i}{2}\partial_x^{-1}\left[\left(e^{2i\tau\partial_x^2}\partial_x^{-2}\overline{f}\right)\left(\partial_x\overline{f}\right)\right] + \frac{i}{2}\partial_x^{-1}\left[\left(\partial_x^{-2}\overline{f}\right)\left(\partial_x\overline{f}\right)\right],$$

$$F_2^\tau(f) = -\frac{i}{2}\partial_x^{-2}e^{i\tau\partial_x^2}\left[\left(e^{i\tau\partial_x^2}\overline{f}\right)\left(e^{-i\tau\partial_x^2}\overline{f}\right)\right] + \frac{i}{2}\partial_x^{-2}\left(\overline{f}^2\right) - \tau\partial_x^{-1}\left[\left(\partial_x\overline{f}\right)\overline{f}\right].$$

Convergence of Scheme II

- Applying Kato-Ponce inequalities, Hardy-Littlewood-Sobolev type inequality, Sobolev embedding theorem, bilinear estimate, we are able to establish

$$\|P_2^\tau(f)\|_r \lesssim \tau^2 \|f\|^2_{\frac{r}{2}+\frac{5}{4}}, \quad r \in (1/2, 7/6]; \quad \|P_2^\tau(f)\|_r \lesssim \tau^2 \|f\|^2_{r+\frac{2}{3}}, \quad r > \frac{7}{6}.$$

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- Convergence of Scheme II obtained by taking $I_0^\tau(f) = 2(F_1^\tau(f) + F_2^\tau(f))$: for $r > 1/2$, $u \in H^{r+p(r)}$, we have $\|u(t_n) - u^n\|_r \lesssim \tau$, where

$$p(r) = \begin{cases} 5/4 - r/2, & 1/2 < r \leq 7/6; \\ 2/3, & r > 7/6. \end{cases}$$

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$$p(r) = \begin{cases} 5/4 - r/2, & 1/2 < r \le 7/6; \\ 2/3, & r > 7/6. \end{cases}$$

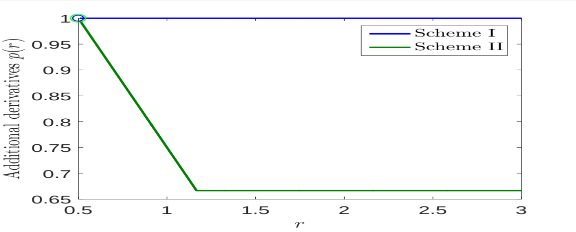

Figure 2: Additional order of regularity required to achieve the first-order accuracy.

First-order scheme III

- Calculation of the term $L_0^\tau(f)$:

$$L_0^\tau(f) = \sum_k \sum_{k_1+k_2=k} \int_0^\tau \mathrm{e}^{is(2k_1k_2-2k^2)} ds \widehat{\overline{f}}_{k_1} \widehat{\overline{f}}_{k_2} \mathrm{e}^{ikx}$$
$$= \tau \sum_k \sum_{k_1+k_2=k} \psi_1(i\tau(2k_1k_2 - 2k^2)) \widehat{\overline{f}}_{k_1} \widehat{\overline{f}}_{k_2} \mathrm{e}^{ikx}.$$

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- Key approximation:

$$\left| \psi_1\left(i\tau(\alpha+\beta)\right) - \psi_1\left(i\tau\alpha\right)\psi_1\left(i\tau\beta\right) - (\mathrm{e}^{i\tau\alpha}-1)\psi_2\left(i\tau\beta\right) + \frac{\mathrm{e}^{i\tau\alpha}-1}{2}\psi_1\left(i\tau\beta\right) \right|$$
$$\lesssim \min\left\{ \frac{|\alpha|^{1+\mu}}{|\beta|}\tau^\mu, \tau^\theta|\alpha|^\theta \right\}, \quad \mu\in[0,1], \quad \theta\in[0,2].$$

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$$\lesssim \min\left\{ \frac{|\alpha|^{1+\mu}}{|\beta|}\tau^\mu, \tau^\theta|\alpha|^\theta \right\}, \quad \mu \in [0,1], \quad \theta \in [0,2].$$

- Convergence of Scheme III : for $r > 1/2$, $u \in H^{r+p(r)}$, we have $\|u(t_n) - u^n\|_r \lesssim \tau$, where

$$p(r) = \begin{cases} 5/4 - r/2, & 1/2 < r < 5/2; \\ 0+, & r = 5/2; \\ 0, & r > 5/2. \end{cases}$$

Convergence of all three first-order schemes

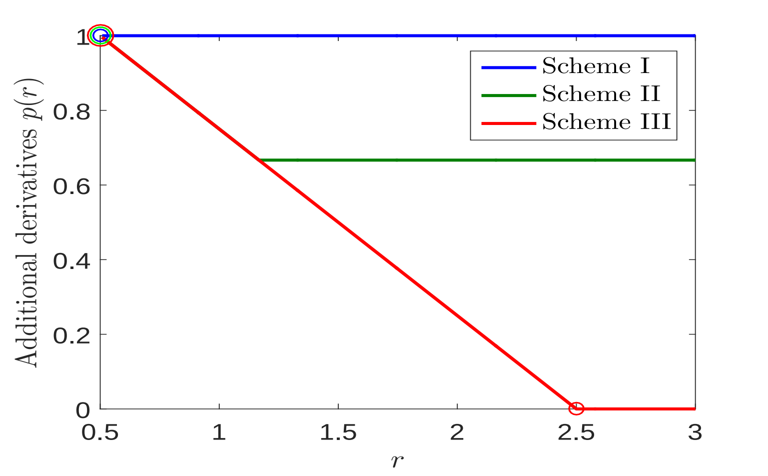

Figure 3: Additional derivatives required to promise the first-order convergence.

Convergence of the first-order methods

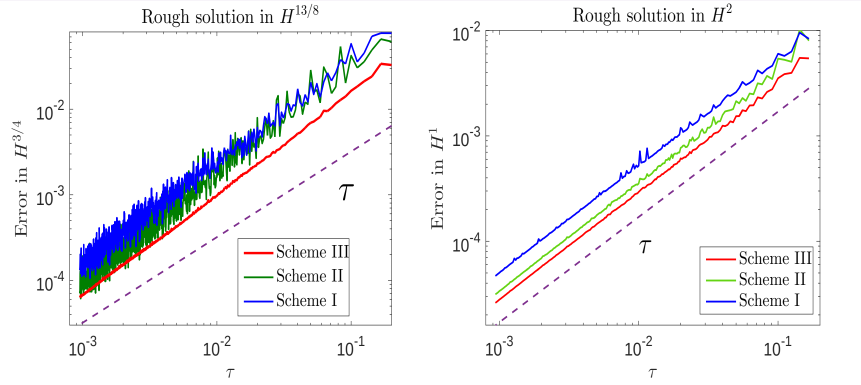

Figure 4: Convergence of the first-order methods for rough solutions.

Thank you!