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One-dimensional hydrodynamic PDE model of turbulent flow with the enstrophy cascade

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Based on the joint works with
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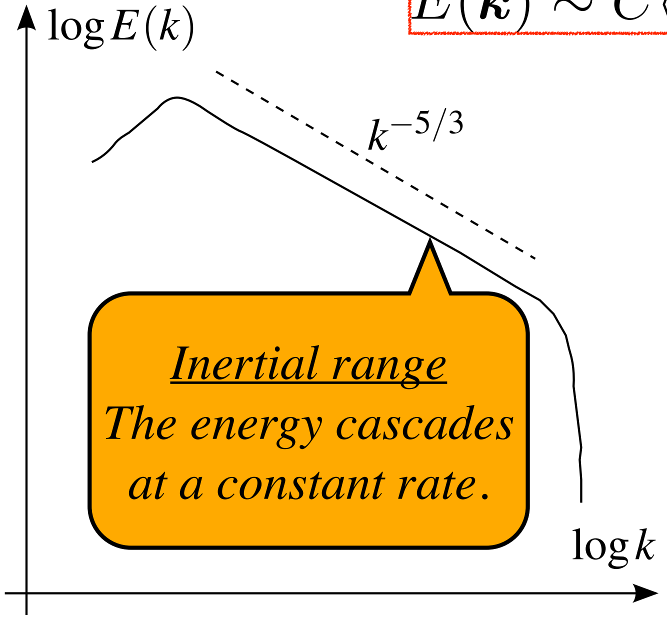
Cascade phenomenon in turbulence

Kolmogorov's Theory of Turbulence

The energy dissipation rate
$$\boxed{\langle \epsilon \rangle := \liminf_{\nu \to 0} \nu \|\nabla u\|^2 > 0}$$

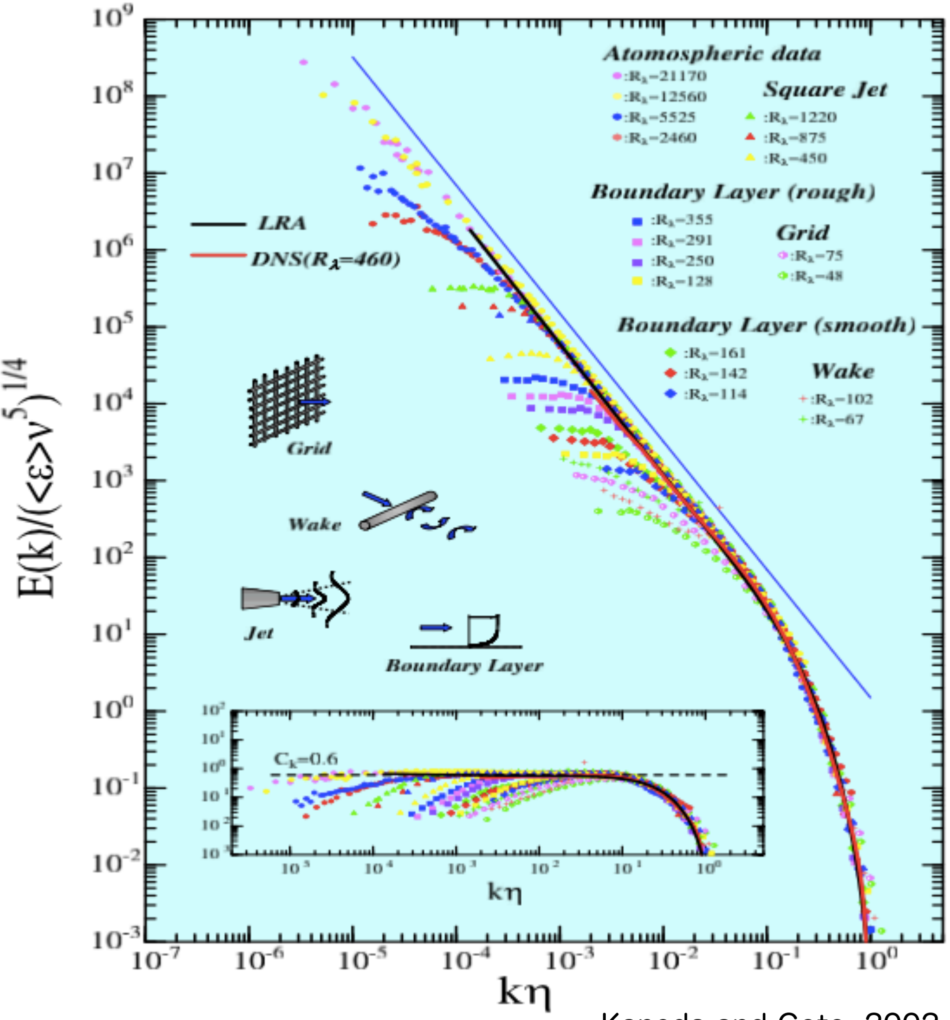
The statistical law of the energy spectra
$$\boxed{E(\boldsymbol{k}) \sim C \langle \epsilon \rangle^{2/3} |\boldsymbol{k}|^{-5/3}}$$



Inertial range
The energy cascades at a constant rate.

Energy is a conserved quantity for inviscid flows, but it suggests that the energy dissipates in the zero-viscous limit, suggesting a singular limit in fluid equations.

Good agreement with many experiments and simulations



Kaneda and Goto, 2002

Describe the cascade phenomena of **the inviscid invariant** (energy, enstrophy, etc.) in terms of solutions of a hydrodynamic equation.

3D Euler equations

$\boldsymbol{u}(\boldsymbol{x},t)$: velocity field $\quad \boldsymbol{\omega}(\boldsymbol{x},t)$: vorticity field $\quad (\boldsymbol{x},t) \in \mathbb{R}^3 \times \mathbb{R}$

Euler equations for the inviscid and incompressible flows:

$$\boxed{\frac{D\omega}{Dt} \equiv \frac{\partial\omega}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{\omega} = \boldsymbol{\omega}\nabla\boldsymbol{u}, \quad \boldsymbol{\omega}(\boldsymbol{x},0) = \boldsymbol{\omega}_0(\boldsymbol{x}) = \nabla\times\boldsymbol{u}_0}$$

Biot-Savart formula: $\quad \boldsymbol{u}(\boldsymbol{x},t) = -\frac{1}{4\pi}\int \frac{\boldsymbol{x}-\boldsymbol{y}}{|\boldsymbol{x}-\boldsymbol{y}|^3} \times \boldsymbol{\omega}(\boldsymbol{y},t)d\boldsymbol{y}$

The quadratic term $\boldsymbol{\omega}\nabla\boldsymbol{u}$ is rewritten by an operator form $\mathcal{D}(\boldsymbol{\omega})\boldsymbol{\omega}$, in which the symmetric part of the matrix $\nabla\boldsymbol{u}$.

$$\mathcal{D} = \frac{1}{2}\left(\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^T\right)$$

Thus we rewrite the Euler equation in a closed form of $\boldsymbol{\omega}$:

$$\boxed{\frac{D\boldsymbol{\omega}}{Dt} = \mathcal{D}(\boldsymbol{\omega})\boldsymbol{\omega}}$$

Constantin-Lax-Majda model

Properties of the operator $\mathcal{D}$:

- It is a singular integral operator.
- It is represented by the convolution of $\boldsymbol{\omega}$ with a kernel homogeneous of degree -3, the spacial dimension.

Hilbert transform: a 1D analogue of the operator $\mathcal{D}(\boldsymbol{\omega})$

$$\boxed{H(\omega) = \frac{1}{\pi}\mathrm{pv}\int_{-\infty}^{\infty} \frac{\omega(y)}{x-y}dy.}$$

The quadratic term $H(\omega)\omega$ is a scalar 1-D analogue of the vortex stretching term $\mathcal{D}(\boldsymbol{\omega})\boldsymbol{\omega}$.

Constantin-Lax-Majda (CLM) equation (1985):

$$\boxed{\partial_t \omega = H(\omega)\omega}$$

P. Constantin, P. D. Lax, and A. Majda. Comm. Pure Appl. Math., Vol. 38, No. 6, pp. 715–724, 1985.

gCLMG model (Okamoto, S-, Wunsch)

Generalized Constantin-Lax-Majda-DeGregorio equation (gCLMG eq.)

$$\partial_t \omega + a v \omega_x - v_x \omega = 0, \quad v_x = H\omega$$

Advection term

Vortex stretching term

$a \in \mathbb{R}$ + Periodic Boundary Condition

(DeGregorio 1990, 1996; Okamoto, S-, Wunsch 2008,2014)

Function spaces:

$$L^2(S^1)/\mathbb{R} = \left\{ f \;\middle|\; f \in L^2(-\pi, \pi), \quad \int_{-\pi}^{\pi} f(x)dx = 0 \right\},$$

$$H^k(S^1)/\mathbb{R} = \left\{ f \;\middle|\; f = \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx), \quad \sum_{n=1}^{\infty}(a_n^2 + b_n^2)n^{2k} < \infty \right\},$$

H. Okamoto, T. Sakajo, and M. Wunsch. Nonlinearity, Vol. 21, pp. 2447–2461, 2008.

Existence of a unique solution

Existence of a unique local solution

Theorem (OSW, 2008) Let $a \in \mathbb{R}$ be given. For all $\omega_0 \in H(S^1)/\mathbb{R}$, there exists a T depending on a and $\|\omega_{0,x}\|_{L^2}$ such that a unique solution

$$\omega \in C^0([0,T]; H^1(S^1)/\mathbb{R}) \cap C^1([0,T]; L^2(S^1)/\mathbb{R})$$

exists with $\omega(x,0) = \omega_0(x)$

A criterion for global existence

Theorem (OSW, 2008) Suppose that $\omega(\cdot,0) \in H^1(S^1)/\mathbb{R}$, that solution exist in $[0,T)$, and that

$$\int_0^T \|H\omega(\cdot,t)\|_{L^\infty} dt < \infty.$$

Then the solution exists in $0 < t < T + \delta$ for some $\delta > 0$.

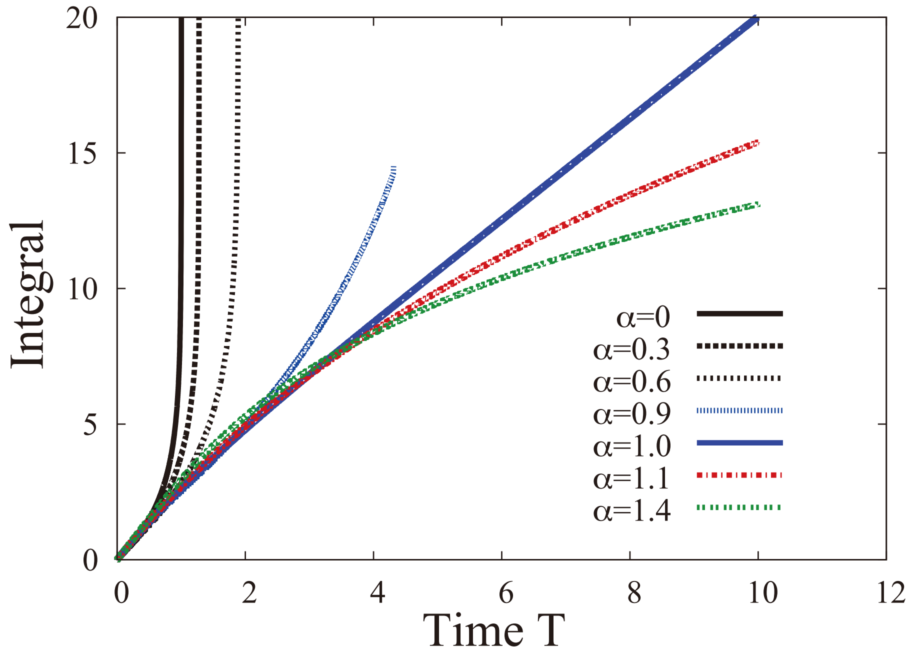
- It is relevant to Beale-Kato-Majda criterion for the 3D Euler eqs.
- It is difficult to prove the criterion for the local solution.

Blow-up or global existence? & Invariant quantity

Plot of $\int_0^T \|H\omega(\cdot,t)\|_\infty dt$

$$\Downarrow$$

$$a_c \sim 0.6$$



Conjecture (OSW, 2014) There exists an $0 < a_c < 1$ such that solutions to gDG eq. exist global in time if $a_c < a < \infty$ (or $a_c \leqq a < \infty$) and that blow-up occurs if $a \leqq a_c$ (or $a < a_c$).

Existence of inviscid invariant quantity

Proposition (OSW, 2008) If $-\infty < a < -1$, then $\|\omega(\cdot,t)\|_{L^p} = \|\omega_0(\cdot)\|_{L^p}$, where $a = -p$.

$a = -2 \Longrightarrow$ the turbulent flow with the cascade of the enstrophy, i.e. $\|\omega(\cdot,t)\|$, is expected.

Existence of global solution

A criterion for global existence *cf*. Beale-Kato-Majda criterion for the 3D Euler eqs.

Theorem (OSW, 2008) Suppose that $\omega(\cdot,0) \in H^1(S^1)/\mathbb{R}$, that solution exist in $[0,T)$, and that

$$\int_0^T \|H\omega(\cdot,t)\|_{L^\infty} dt < \infty.$$

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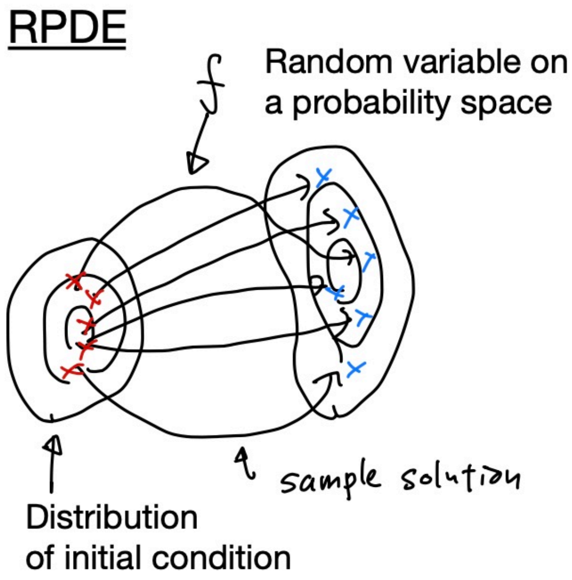
A hydrodynamic model for turbulence

Viscous term + Random forcing ν: the (model) viscous coefficient

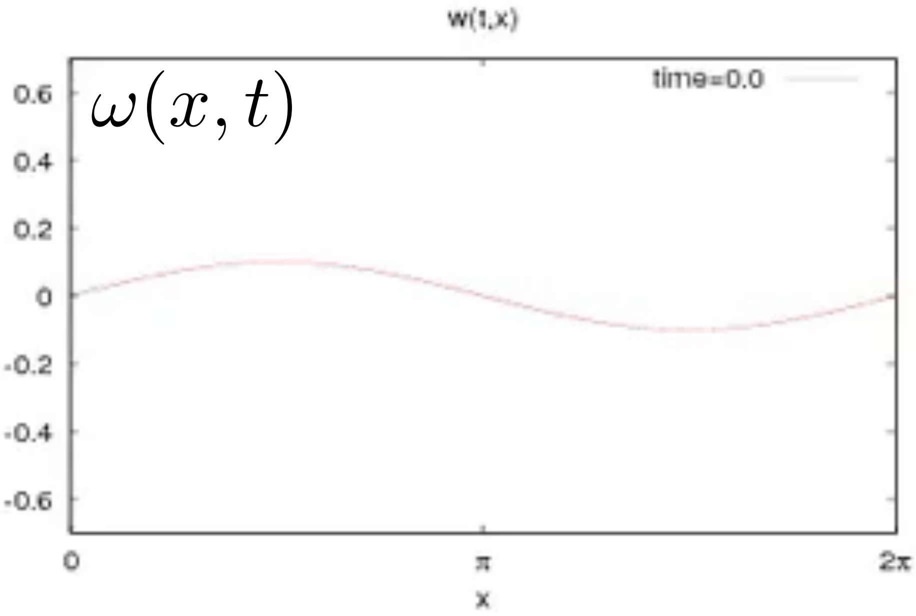
$$\boxed{\partial_t \omega + a v \omega_x - v_x \omega = \nu \omega_{xx} + f, \quad \nu > 0}$$

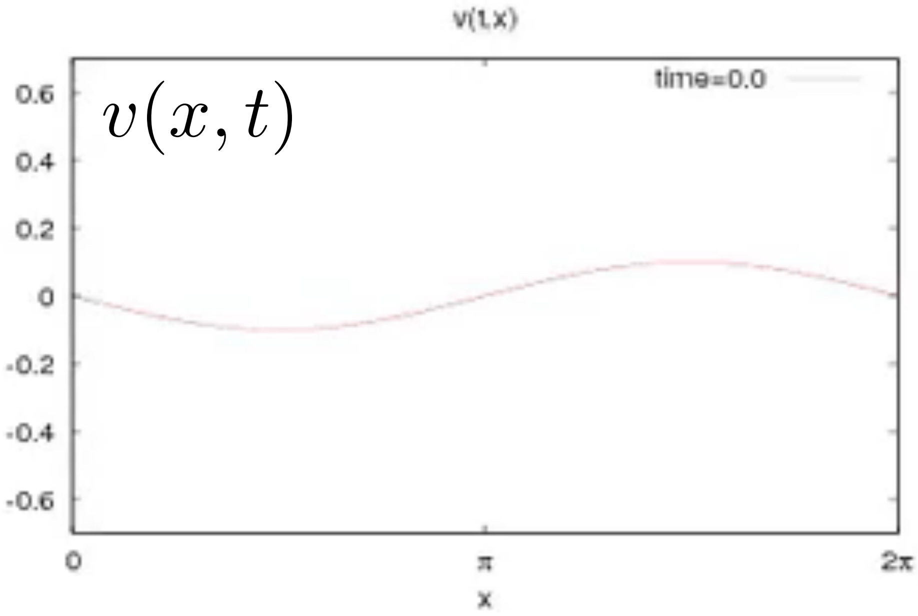
Two Choices of random forcing

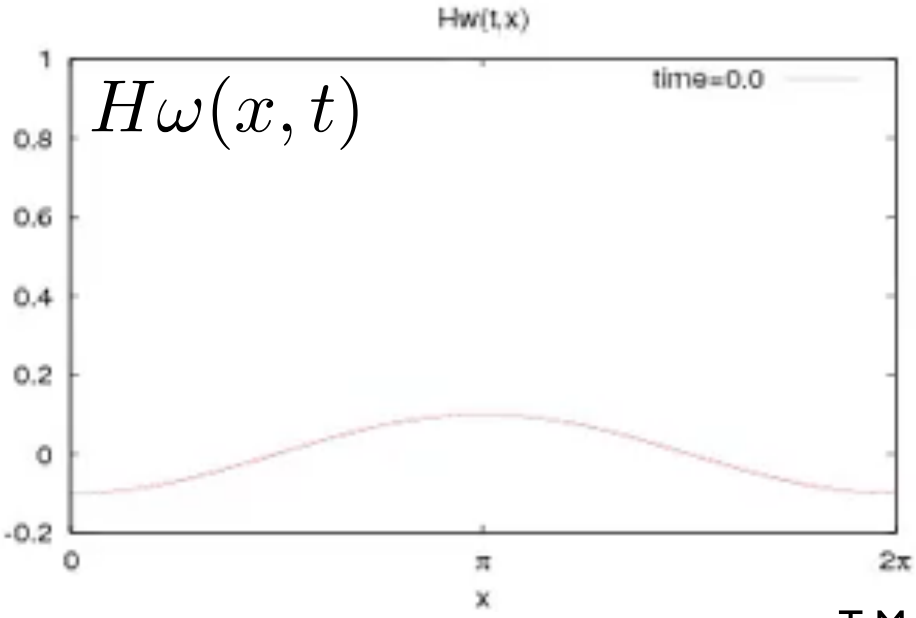
- A Winer process, whose Fourier coefficient $f(k,t)$ with the large-scale wavenumbers $k = \pm 1$ are set to Gaussian, δ-correlated-in-time, and independent random variables with zero mean. $\Longrightarrow$ Stochastic PDE.
- The forcing functions f are regarded as random variables defined on a certain probability space Ω. $\Longrightarrow$ Random PDE.



SPDE: Evolution of a solution (a=-2)







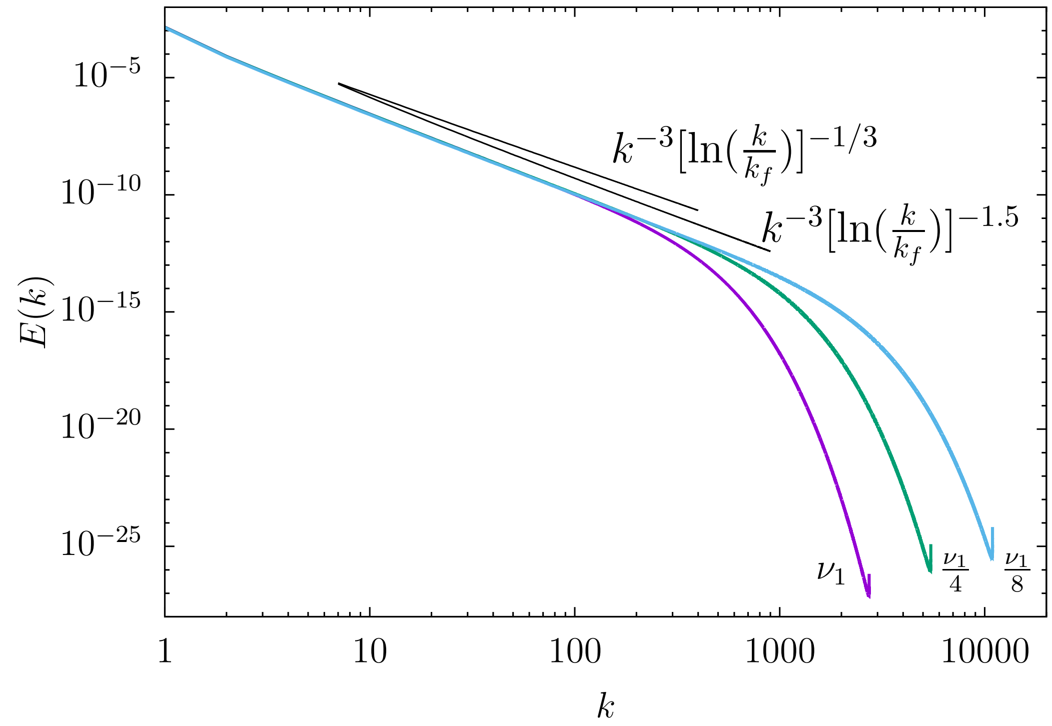
$$\nu_1 = 2.5 \times 10^{-5}$$

- ◆ L^2 norm is statistically stationary
- ◆ Appearance/disappearance of moving sharp spikes (singularities)
- ◆ Check if the scaling laws of enstrophy cascade is observed

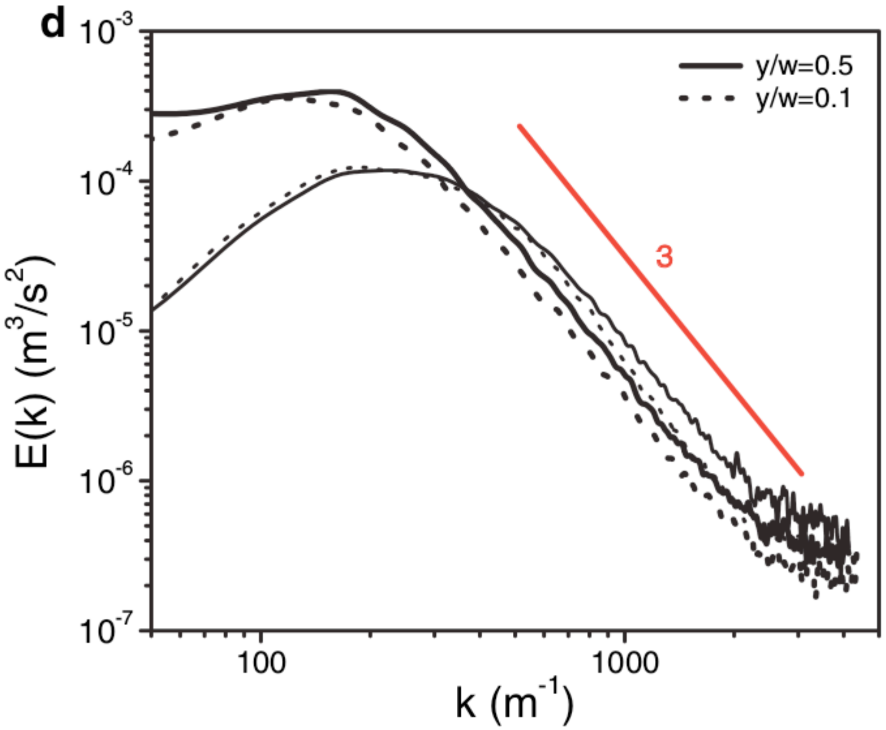
T. Matsumoto and T. Sakajo. Phys Rev E, Vol. 93, p. 053101, 2016.

SPDE: Energy (Time averaged)

$$\boxed{E(k) = \sum_{k \leqq |k'| \leqq k+\Delta k} \frac{1}{2} |\hat{u}(k,t)|^2}$$



$$\nu_1 = 2.5 \times 10^{-5}$$

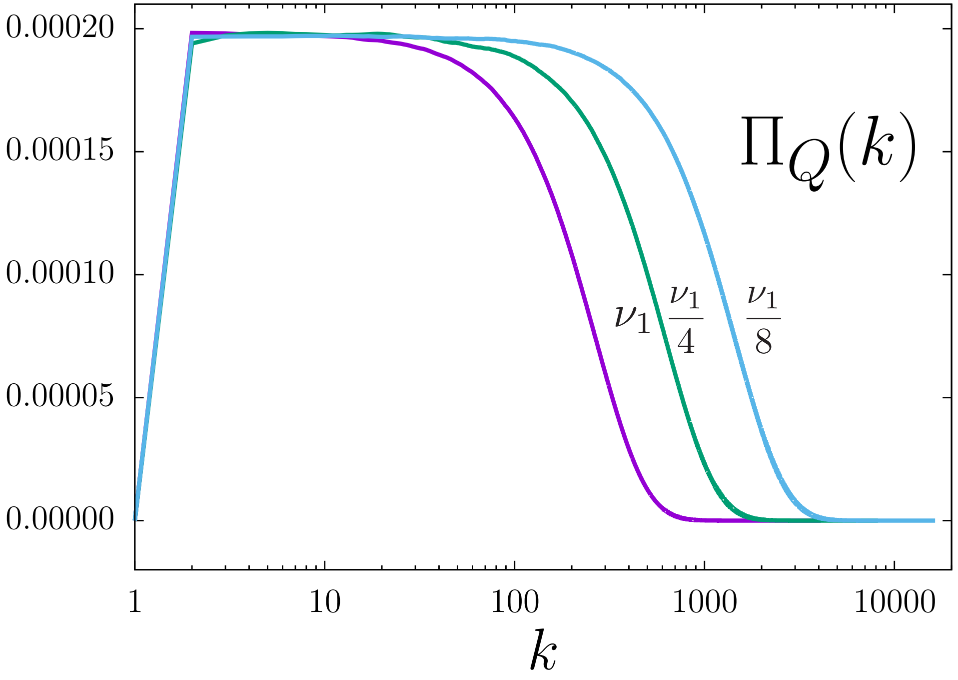


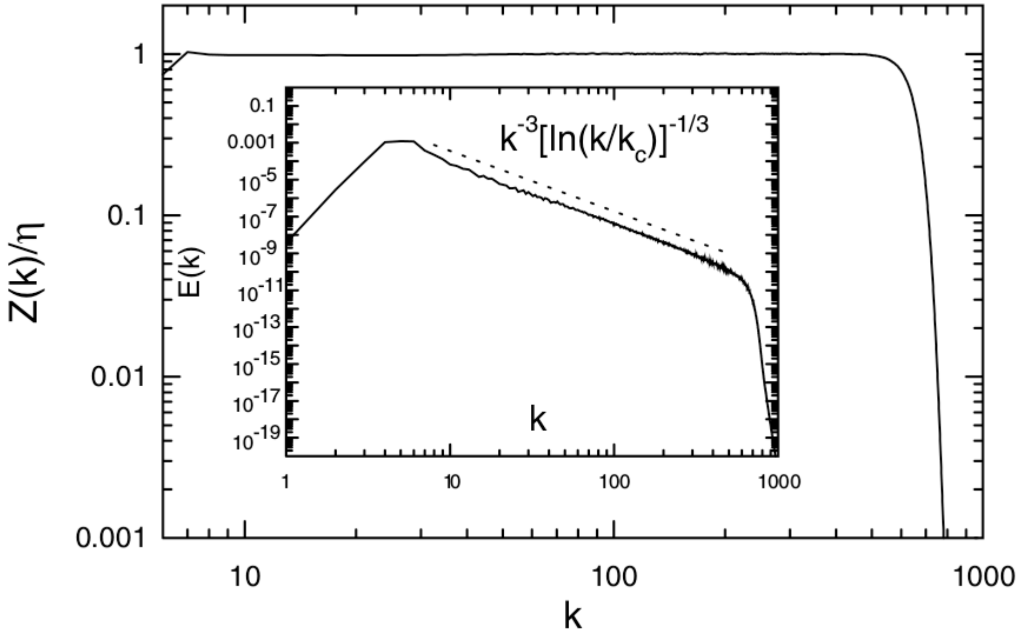
2D enstrophy cascade turbulence, soap-film experiments (Tran et al. 2012)

- The inertial range appears and it expands as ν→0.
- A deviation from k^{-3} (cf. Kraichnan-Leith-Batchelor theory)

SPDE: Enstrophy flux

$$\boxed{\Pi_Q(k,t) = \sum_{l\geq k}\sum_{|k'|=l}\sum_{p+q=k'} \mathrm{Im}[\hat{\omega}(k',t)(aq-p)\hat{u}(p,t)\hat{u}(q,t)]}$$





2D enstrophy cascade turbulence,

Simulation Chen et al. 2003:

- ◆ 👍 A plateau region (constant enstrophy flux) is observed.
- ◆ 👍 Numerical evidence of enstrophy cascade.
- ◆ (NG) It is difficult to compute higher-order statistics for SPDE
- ◆ (NG) Mathematical analysis is not easy. (Existence of invariant measure, 2023)

T. Matsumoto & T. Sakajo, PRE 2016

RPDE: random gCLMG eq (S.-, Tsuji '23)

The gCLMG equation with random forcing function on a probability space

$$\omega_t + av\omega_x - v_x\omega = \nu\omega_{xx} + f, \quad v_x = H(\omega)$$
$$\omega(0,x) = \omega_0(x), \quad \text{periodic boundary condition on } \mathbb{S}^1 = \mathbb{R}/(2\pi\mathbb{Z}).$$

Purpose:

- Global well-posedness of the viscous gCLMG equation with (deterministic) forcing functions.
- Existence of a stochastic process $\omega(t)$ for random initial data and random forcing functions.
- Investigate statistical properties of solutions using the Galerkin approximation with generalized Polynomial Chaos (gPC).

Function space:

X_T^m: the set of continuous functions from $[0,T]$ to $\dot{H}^m$.

$$\|u\|_{X_T^m} := \sup_{0 \leqq t \leqq T} \|u(t)\|_{\dot{H}^m} \qquad \dot{H}^m := \left\{ u \in H^m(\mathbb{S}^1) \;\middle|\; \int_0^{2\pi} u(x)dx = 0 \right\}$$

Y. Tsuji and T. Sakajo. Nonlinearity to be published, 2023

Definitions of solutions

Definition. Let $m \in \mathbb{N}$ and $0 < T < \infty$.

- For the initial data $\omega_0 \in \dot{H}^m$ and the forcing function $f \in X_T^m$, $\omega \in X_T^m$ is called the mild solution to the gCLMG equation, if

  $$\omega(t) = e^{\nu t \triangle}\omega_0 + \int_0^t e^{\nu(t-s)\triangle}\left\{-a(v\omega)_x(s) + (1+a)(u_x\omega)(s) + f(s)\right\} ds$$

  holds in $\dot{H}^m$ for $t \in [0,T]$, where $e^{\nu t \triangle} = \mathcal{F}^{-1}e^{-t\nu n^2}\mathcal{F}$ for $t \geqq 0$ represents the heat semi-group.

- For the initial data $\omega_0 \in \dot{H}^{m+2}$ and the forcing function $f \in X_T^{m+2}$, we call $\omega \in C([0,T];\dot{H}^m) \cap C^1((0,T];\dot{H}^m) \cap C((0,T];\dot{H}^{m+2})$ is the strong solution, if the gCLMG equation holds in $\dot{H}^m$.

- For the initial data $\omega_0 \in \dot{H}^m$ and the forcing function $f \in X_\infty^m, \omega \in X_\infty^m$ is said to be the global mild solution, if $\omega|_{[0,T]} \in X_T^m$ for any $0 < T < \infty$ is the mild solution to the gCLMG equation for the initial data $\omega_0 \in \dot{H}^m$ and the forcing function $f|_{[0,T]} \in X_T^m$.

Mathematical Results

☐ Existence of a unique mild solution

Theorem. Let $a \in \mathbb{R}$, $\nu > 0$ and $m \in \mathbb{N}$. Suppose that $f \in X^m_\infty$ and $\omega_0 \in \dot{H}^m$. Then, there exists $T > 0$ such that the gCLMG equation has a unique mild solution $\omega \in X^m_T$.

☐ Continuity of solution with respect to the initial data and the forcing

Theorem. Let $0 < T < \infty$, $a \in \mathbb{R}$, $\nu > 0$ and $m \in \mathbb{N}$. Suppose that $f_1, f_2 \in X^m_T$ and $\omega_{01}, \omega_{02} \in \dot{H}^m$. Suppose that $\omega_i \in X^m_T$, $i = 1, 2$ represents the mild solution of the gCLMG equation for the forcing function $f_i \in X^m_T$ and the initial data $\omega_{0i} \in \dot{H}^m$. Then, there exists a constant $C(a, \nu, T, \|\omega_1\|_{X^m_T}, \|\omega_2\|_{X^m_T}) > 0$ such that the following inequality holds.

$$\|\omega_1 - \omega_2\|_{X^m_T} \leqq C(\|f_1 - f_2\|_{X^m_T} + \|\omega_{01} - \omega_{02}\|_{\dot{H}^m}).$$

☐ A priori estimate (the solution remains bounded)

Lemma. Let $a = -2$, $\nu > 0$, $m \in \mathbb{N}$, $f \in X^m_\infty$ and $\omega_0 \in \dot{H}^m$. Suppose that there exists a classical solution $\omega \in C^1([0, T]; \dot{H}^m) \cap C([0, T]; \dot{H}^{m+2})$ to the gCLMG equation for any $T > 0$. Then the solution ω satisfies the following estimate.

$$\|\omega\|^2_{X^m_T} \leqq C(m, \nu, T)(P_m(\|\omega_0\|^2_{\dot{H}^m}) + Q_m(\|f\|^2_{X^m_T})),$$

where $P_m(x)$ and $Q_m(x)$ denote polynomials of degree $3m$ having non-negative coefficients that are independent of ν, T, ω_0 and f.

Mathematical Results

- **Existence of a unique global solution**

Theorem. Let $a = -2$, $\nu > 0$ and $m \in \mathbb{N}$. Suppose the forcing functon $f \in X^m_\infty$ and the initial data $\omega_0 \in \dot{H}^m$. Then there exists a unique mild solution $\omega \in X^m_\infty$ to the gCLMG equation globally in time. Moreover, for any $T > 0$, the solution satisfies the following estimate.

$$\|\omega\|_{X^m_T} \leqq C(m,\nu)(P_m(\|\omega_0\|^2_{\dot{H}^m}) + Q_m(\|f\|^2_{X^m_T})),$$

where P_m and Q_m are polynomials of degree $3m$ with non-negative coefficients.

- **Existence of a unique global stochastic process**

Theorem. Let $a = -2$, $\nu > 0$ and $m \in \mathbb{N}$. For a given probability space $(\Omega, \mathcal{F}, P)$, we introduce random variables $f : \Omega \to X^m_\infty$ and $\omega_0 : \Omega \to \dot{H}^m$ saisfying $f \in \cap_{p=1}^\infty L^p(\Omega; X^m_T)$ and $\omega_0 \in \cap_{p=1}^\infty L^p(\Omega; \dot{H}^m)$ for any $T > 0$. Then there exists a stochastic process $\omega : \Omega \to X^m_\infty$ uniequly such that $\omega|_{[0,T]} \in L^2(\Omega; X^m_T)$ for $0 < T < \infty$, and for any $\eta \in \Omega$, $\omega^\eta = \omega(\eta) \in X^m_\infty$ is a mild solution to the gGLMG equation.

We are going to compute the stochastic process numerically to observe its statistical properties.

Galerkin approximation

Random variable

$Z : \Omega \to \mathbb{R}^d$: random variable

$\widetilde{f} : \mathbb{R}^d \to \dot{H}^m$: measurable function

$$\Longrightarrow f(\eta) = \widetilde{f}(Z(\eta)) \text{ for } \eta \in \Omega.$$

$(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), P^Z)$: the space of the pushforward measure of Z.

$\Longrightarrow$ The global mild solution $\omega(t) \in L^2_{P^Z}(\mathbb{R}^d; \dot{H}^m)$ exists for any $t \geqq 0$

The Galerkin approximation (the gPC expansion)

The pseudo spectral approximation in $\dot{H}_m$: $\{e_n(x) = \mathrm{e}^{inx}/2\pi\}$ ←Fourier series

The generalized Polynomical Chaos expanstion in $L^2_{P^Z}$: $\{\Phi_m(Z)\}$ ←Orthogonal polynomials

The projection $P_{N,M} := L^2(\Omega; \dot{H}^m) \to L^2(\Omega; \dot{H}^m)$ of the function $\omega(t, x, \eta) \in L^2(\Omega; \dot{H}^m)$ is given by

$$\omega^{N,M}(t, x, \eta) := P_{N,M}\omega(t, x, \eta) = \sum_{m=0}^{M} \sum_{|n| \leqq N} \widehat{\omega}(t, n, m)\mathrm{e}^{inx}\Phi_m(Z(\eta)),$$

where

$$\widehat{\omega}(t, n, m) := \frac{\mathbb{E}_{P^Z}[\langle \omega(t, \cdot, \cdot), e_n(\cdot)\rangle_{L^2}\Phi_m(\cdot)]}{\mathbb{E}_{P^Z}[\Phi_m^2]}.$$

Computation of Averages

The gPC expansion of the solution

$$\omega(t,x,\eta) \approx \sum_{m=0}^{M} \sum_{n=-N}^{N} \widehat{\omega}(t,n,m) e^{inx} \Phi_m(Z(\eta))$$

The average of the solution

$$\mathbb{E}[\omega](t,x) = \sum_{n=-N}^{N} \widehat{\omega}(t,n,0) \mathrm{e}^{inx} \mathbb{E}_{PZ}[\Phi_0]$$

The average of the enstrophy spectra

$$\mathbb{E}[\|\omega\|_{L^2}^2](t,k) = \frac{1}{2} \sum_{m=0}^{M} \sum_{|\ell|=k,k+1} |\widehat{\omega}(t,\ell,m)|^2 \mathbb{E}_{PZ}[\Phi_m^2]$$

The average of the p-th moment

$$M_p[\omega](t,r) = \mathbb{E}[|\omega(t,r,\cdot)|^p] \quad \widetilde{\omega}(t,r,m) := \frac{\mathbb{E}_{PZ}[\omega(t,r,\cdot)\Phi_m(\cdot)]}{\mathbb{E}_{PZ}[\Phi_m^2]} = \sum_{n=-N}^{N} \widehat{\omega}(t,n,m) \mathrm{e}^{inr}$$

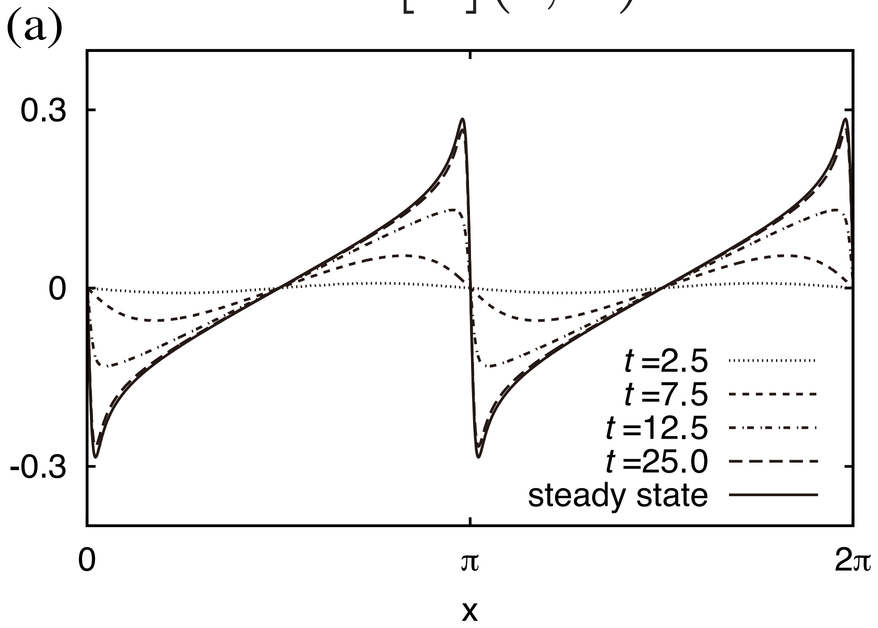
$$M_2[\omega](t,r) = \sum_{m_1,m_2=0}^{M} \widetilde{\omega}(t,r,m_1) \overline{\widetilde{\omega}(t,r,m_2)} \mathbb{E}_{PZ}[\Phi_{m_1}\Phi_{m_2}]$$

$$M_4[\omega](t,r) = \sum_{m_1,m_2,m_3,m_4=0}^{M} \widetilde{\omega}(t,r,m_1) \overline{\widetilde{\omega}(t,r,m_2)} \widetilde{\omega}(t,r,m_3) \overline{\widetilde{\omega}(t,r,m_4)} \mathbb{E}_{PZ}[\Phi_{m_1}\Phi_{m_2}\Phi_{m_3}\Phi_{m_4}]$$

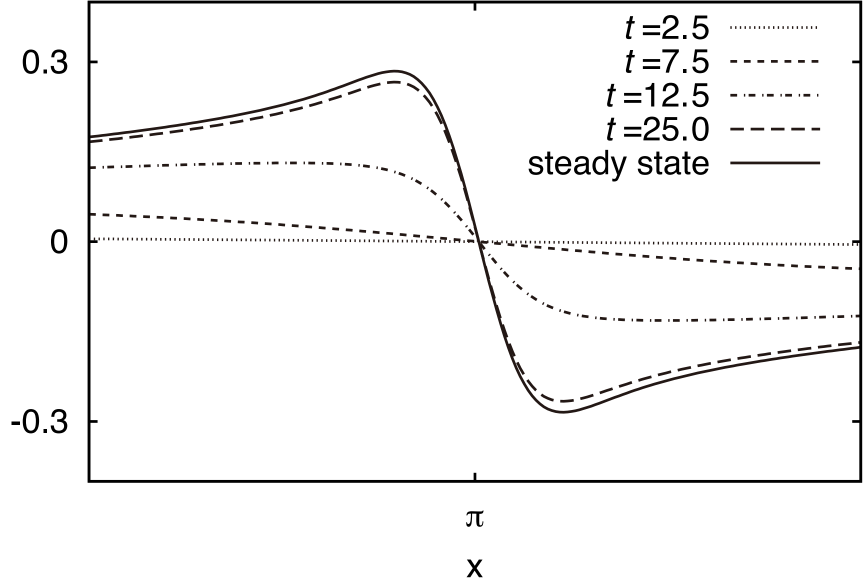
A single numerical computation yields the statistical property of the distribution!

Evolution of the average

$\mathbb{E}[\omega](t,x)$



(b) **Close-up near the pulse**



Random forcing

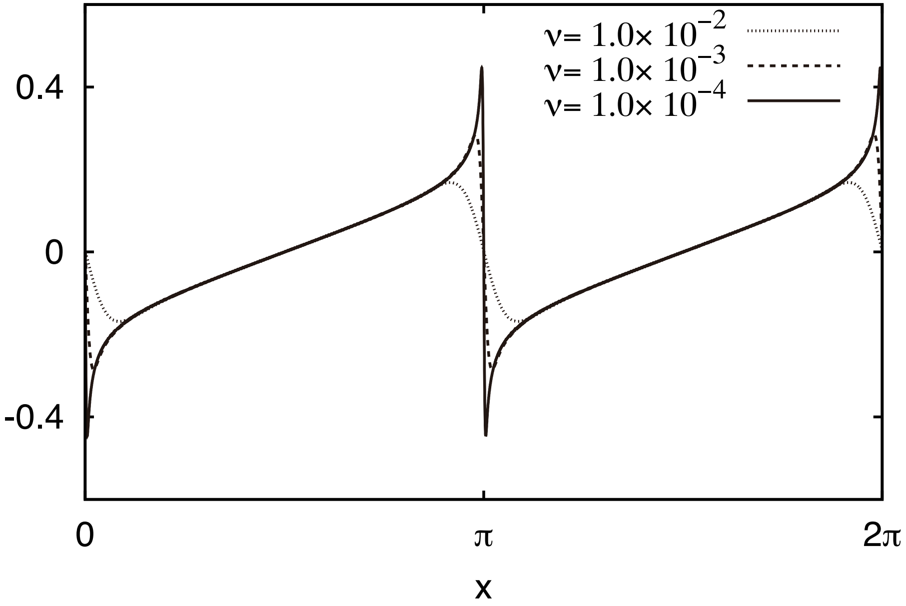
- $f^\eta(t,x) = 0.01 \times (2Z(\eta)-1)\sin x.$
- $Z(\eta) \sim$ the uniform distribution on $[0,1]$.
- $\Phi_m(Z)$: the Legendre polynomials.

The evolution

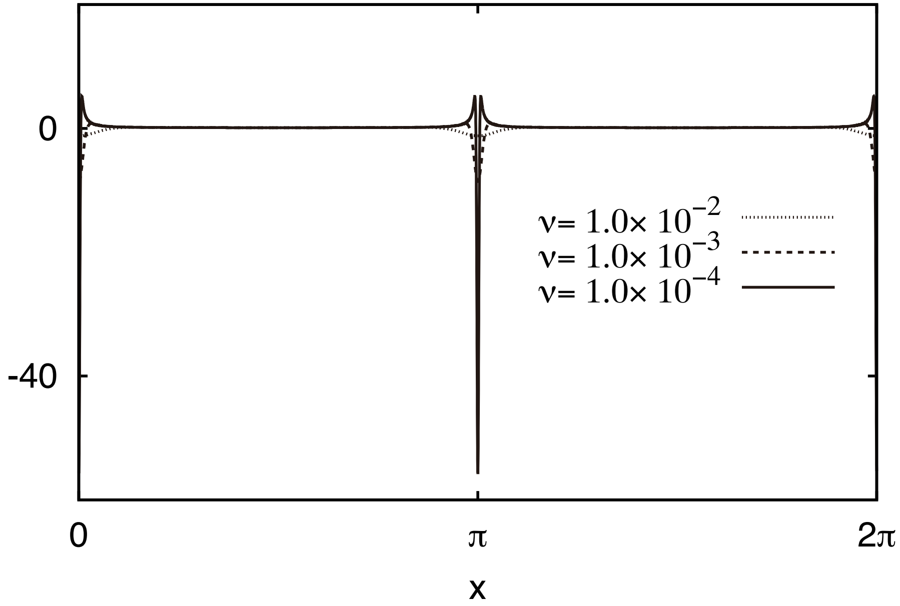
- The viscous coefficient: $\nu = 1.0 \times 10^{-3}$.
- The solution tends to be a stationary state with two peaks at $x=0$ and π.
- It is an invariant distribution of $\mathcal{M}_t^\infty$.
- The solution at $T_s = 237.5$ is used as the steady state to compute statistical quantities.

Average of Solutions for various ν

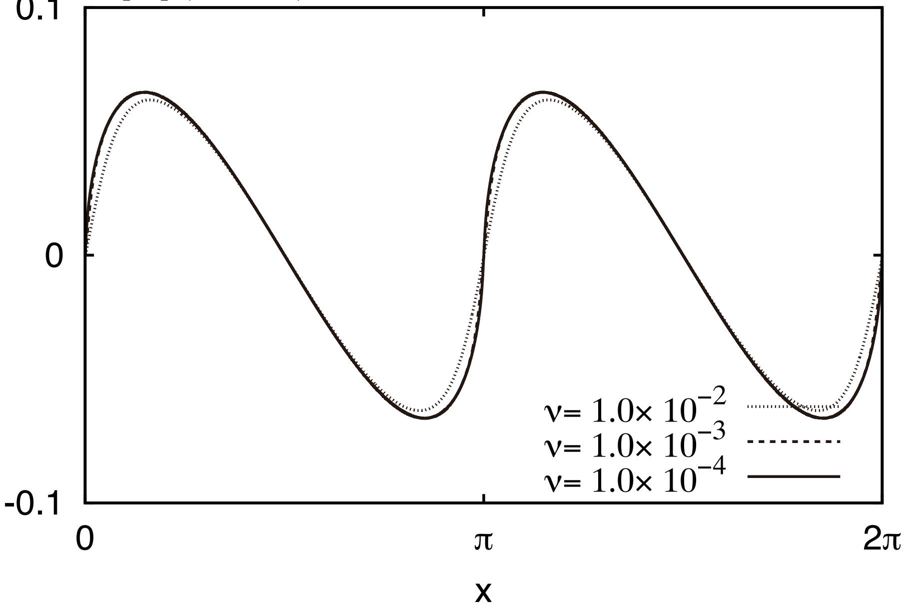
(a) $\mathbb{E}[\omega](T_s, x)$



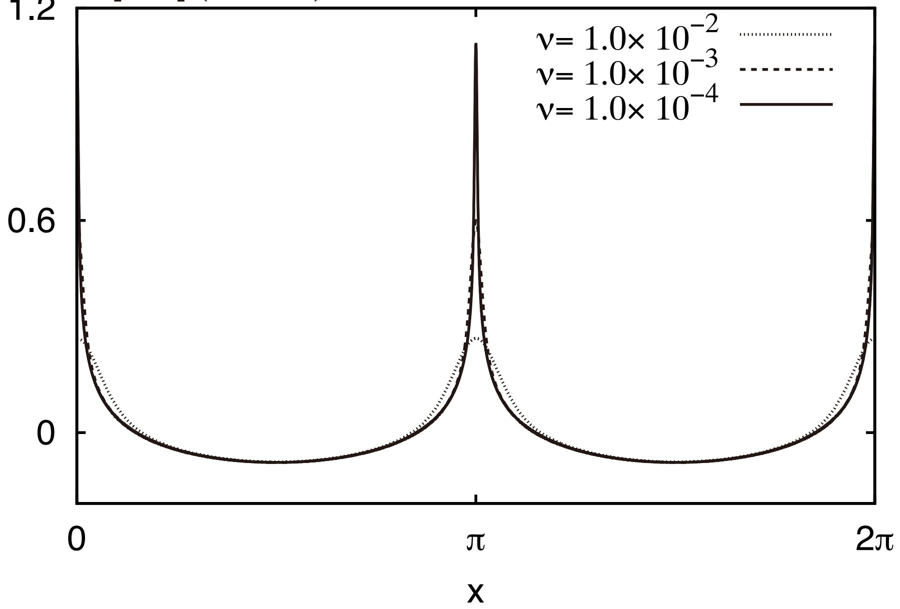
(b) $\mathbb{E}[\omega_x](T_s, x)$



(c) $\mathbb{E}[u](T_s, x)$

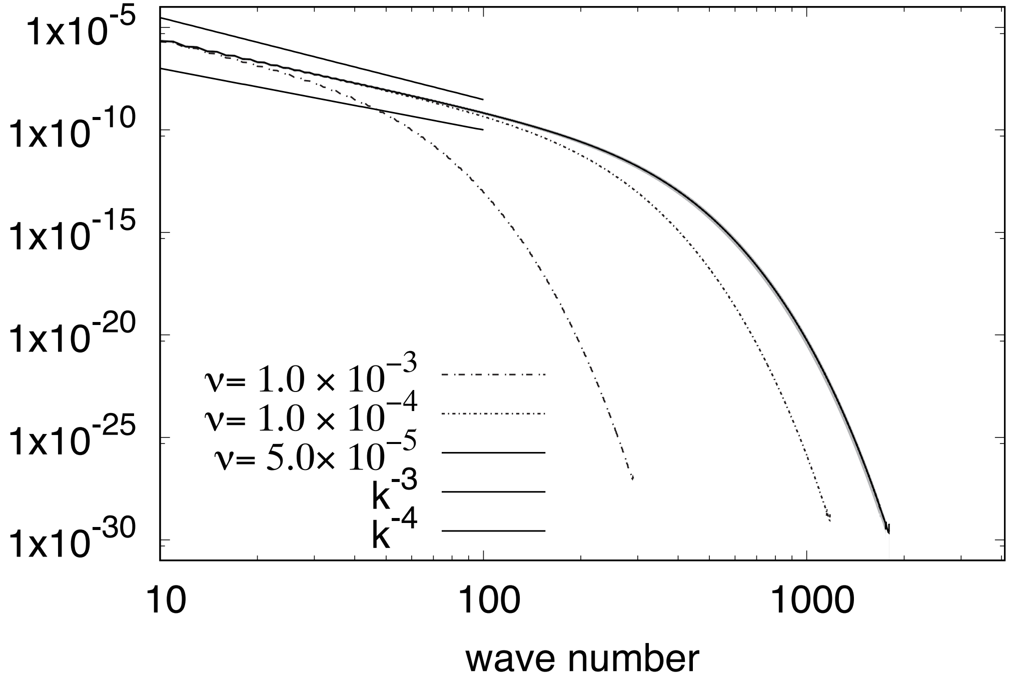


(d) $\mathbb{E}[u_x](T_s, x)$



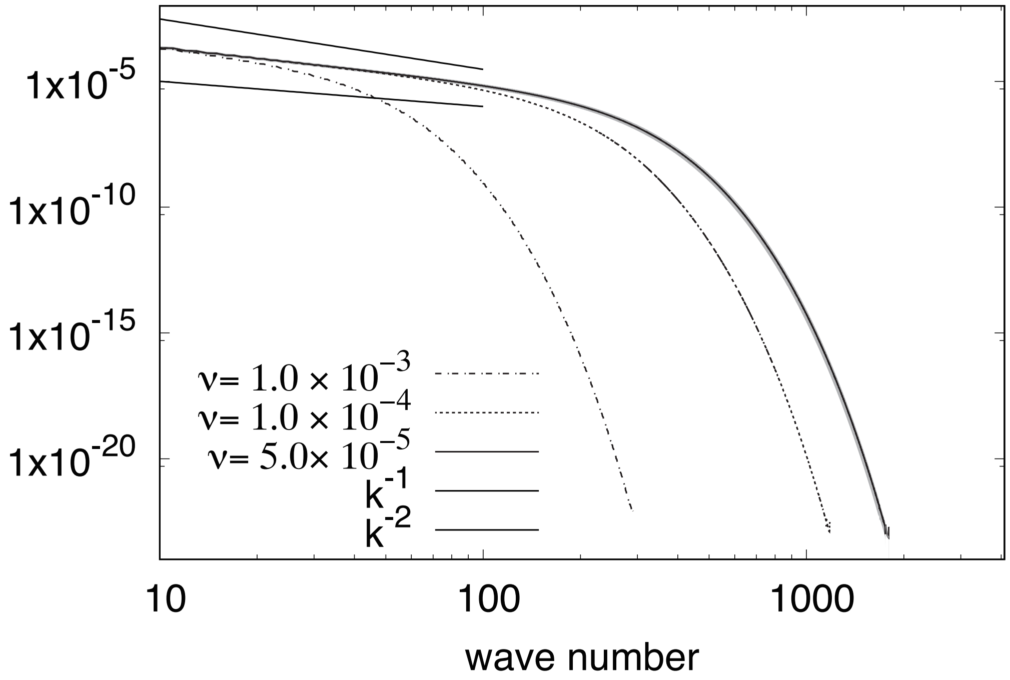
Energy and enstrophy spectra

(a) $\mathbb{E}[\|u\|^2_{L^2}](T_s, k)$



- The average of the energy spectra.
- The decay rate in the inertial range lies in the range of k^{-3} and k^{-4}.
- The dimensional analysis: $\langle \widehat{u}(k) \rangle \simeq k^{-3}$.

(b) $\mathbb{E}[\|\omega\|^2_{L^2}](T_s, k)$



- The average of the enstrophy spectra.
- The decay rate in the inertial range lies between k^{-1} and k^{-2}.
- The dimensional analysis: $\langle \widehat{\omega}(k) \rangle \simeq k^{-1}$.

Good agreement with the scaling laws of the energy spectra for SPDE

Structure Functions

Structure function $$S_p[u](r) := \langle (u(t,x+r) - u(t,x))^p \rangle$$

↑**3D turbulence : isotropic, homogeneous, statistically steady**

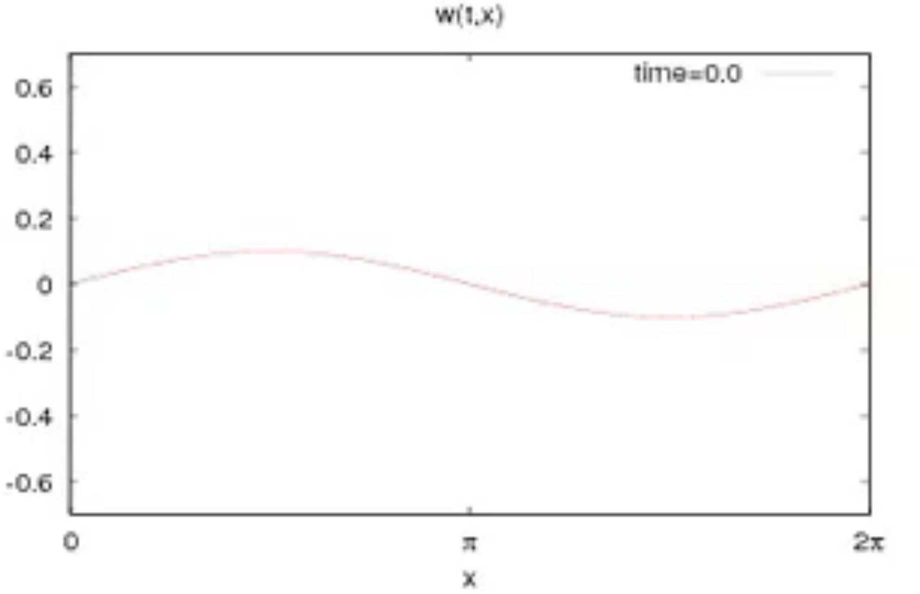
Structure functions for the gCLMG equation

Local *p*-th order structure function:

$$\mathcal{S}_p[\omega](t,x,r) = \mathbb{E}[|\omega^\eta(t,x+r) - \omega^\eta(t,x)|^p]$$

$$\mathcal{S}_p[u](t,x,r) = \mathbb{E}[|u^\eta(t,x+r) - u^\eta(t,x)|^p]$$

- The steady distribution
- The pulse center wonders uniformly



The *p*-th order structure function:

$$S_p[\omega](r) = \mathbb{E}_x[\mathcal{S}_p[\omega](T_s,\cdot,r)]$$

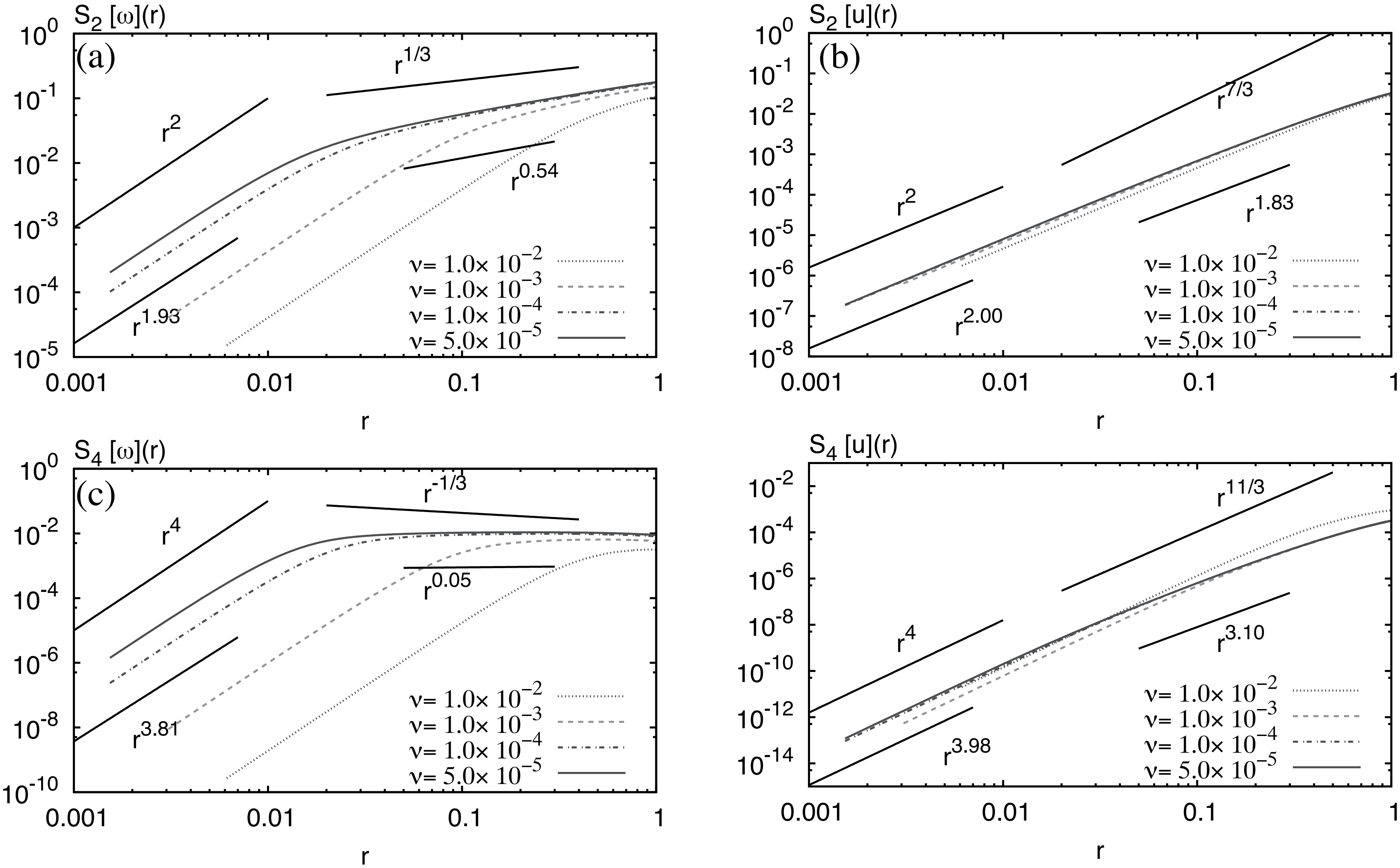
$$= \int_0^{2\pi} \mathcal{S}_p[\omega](T_s,x,r)dx \approx \frac{2\pi}{N}\sum_{n=0}^{N-1} \mathcal{S}_p[\omega](T_s,x_n,r),$$

Dimensional analysis

$$S_p[\omega](r) \simeq r^p,\ S_p[u](r) \simeq r^p \ \ (r \ll 1)$$

$$S_p[\omega](r) \simeq r^{(3-p)/3},\ S_p[u](r) \simeq r^{(3+2p)/3} \ \ (r \approx 1)$$

Structure Functions



The scaling laws deviate from the dimensional analysis, showing intermittency

Intermittency and singular limit

Theorem by Fricsch U. Frisch. Turbulence. The Legacy of A. N. Kolmogorov. Cambridge University Press, 1996.

Suppose that

- the structure function of even order for the flow velocity v satisfies $S_{2p}[v](r) \sim r^{\zeta_{2p}}$ over the inertial range
- the inertial range extends with $\nu \to 0$
- for a certain $p \in \mathbb{N}$, the two consecutive exponents satisfies $\zeta_{2p} > \zeta_{2p+2}$.

Then the maximum velocity diverges as $\nu \to 0$.

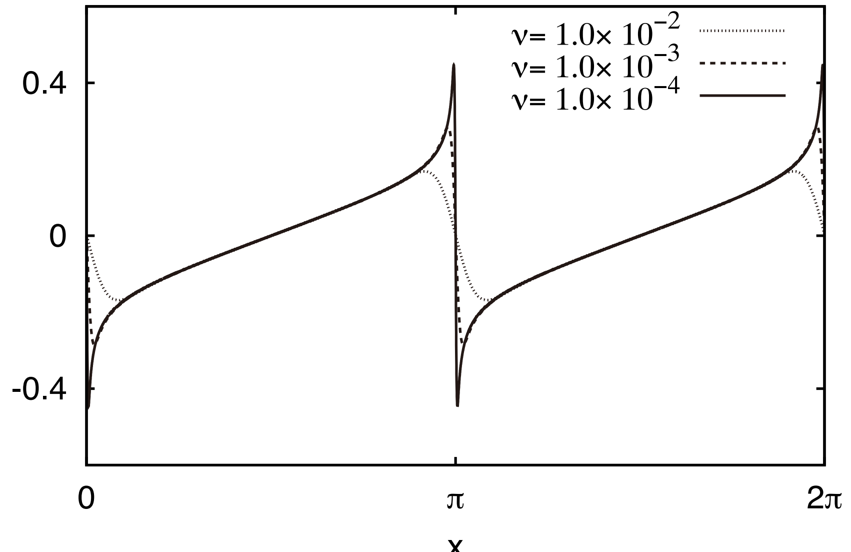
The numerical computation indicates that … The vorticity function diverges as ν→0.

$$S_2[\omega](r) \sim r^{\zeta_2},$$
$$S_4[\omega](r) \sim r^{\zeta_4}$$
$$\zeta_2 = 0.54 > \zeta_4 = 0.05$$

+The inertial range expands

$$\Longrightarrow$$

$\mathbb{E}[\omega](T_s^\nu, x)$



Summary

- ☑ The gCLMDG equation is an interesting one-dimensional mathematical model bringing us useful insights into the balance of nonlinear and linear terms in fluid equations, providing a one-dimensional hydrodynamic model for “turbulent” flow with the cascade of inviscid invariants.
- ☑ SPDE: The turbulent flow is generated by a randomly moving pulse with sharp peaks, yielding the cascade of the enstrophy (the inviscid conserved quantity).
- ☑ RPDE: We have shown mathematically the existence of a stochastic process that is defined from the global solution to the gCLMG equation with uniformly distributed random forcing.
- ☑ Numerical computations of the stochastic process indicate the existence of a steady distribution of solutions with the enstrophy and energy cascades relevant to the pulse turbulence. We find the statistical laws of the structure functions with intermittency.
- ☑ Future work: Mathematical analysis of the steady distribution.

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