Direct and inverse cascades in BEC Wave Turbulence

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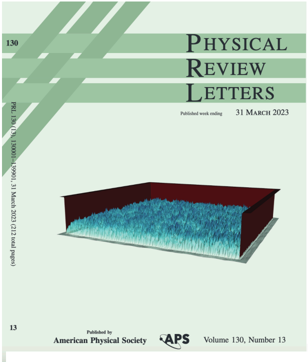
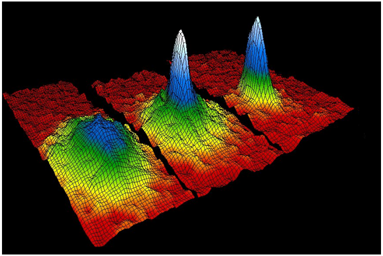


Vladimir Zakharov 1939 - 2023

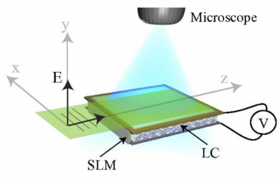
Founding father of Wave Turbulence, Theory of Integrability and Solitons and many other subjects.

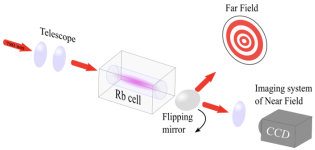
Bose-Einstein Condensate

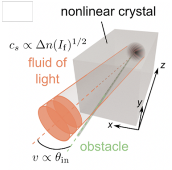


In 1995, researchers from JILA (NIST/U. of Colorado) create a new state of matter predicted in 1920's by Einstein and Bose. Cooling rubidium atoms to $< 170\, nK$ caused the individual atoms condense into a coherent state. The graphic shows successive density snap shots.

Nonlinear Optical Systems at INPHYNI







Left: Liquid Crystal Cell for 1D turbulence (Bortolozzo & Residori). Centre, right: Hot vapour (R. Kaizer) and photorefractive crystal (M. Bellec & C. Michel) for 2D turbulence.

BEC and Optical turbulence.

BEC is described by Gross-Pitaevskii (a.k.a. NLS) equation:

$$i \frac{\partial \psi}{\partial t} + \nabla^2 \psi - |\psi|^2 \psi = 0. \tag{1}$$

where ψ is a complex scalar field.
GP equation (1) conserves two quantities with positive quadratic parts—the energy and the total number of particles,

$$N = \int |\psi(\mathbf{x}, t)|^2 d\mathbf{x}\,, \tag{2}$$

and the total energy,

$$H = \int \left[|\nabla \psi(\mathbf{x}, t)|^2 + \frac{1}{2} |\psi(\mathbf{x}, t)|^4 \right] d\mathbf{x}\,, \tag{3}$$

Fluid properties of the GP system: Madelung transformation

What makes the GP system to act as a fluid? There are strong parallels and analogies between the GP model and the the classical fluids, which could be understood by making the following change of variables called the Madelung transformation: $\psi = \sqrt{\rho}\, e^{i\phi}$. After this transformation, the GP equation (1) becomes very similar to the Euler equations for an ideal fluid with density ρ and velocity $\mathbf{u} = 2\nabla\phi$:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0, \tag{4}$$

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\, \mathbf{u} = -\frac{\nabla \rho^2}{\rho} + \nabla \left(2 \frac{\nabla^2 \sqrt{\rho}}{\sqrt{\rho}} \right). \tag{5}$$

Hence the GP system possesses fluid-like states including randomly moving vortices and waves, i.e. vortex and wave turbulence.

Important structures in hydro and BEC turbulence

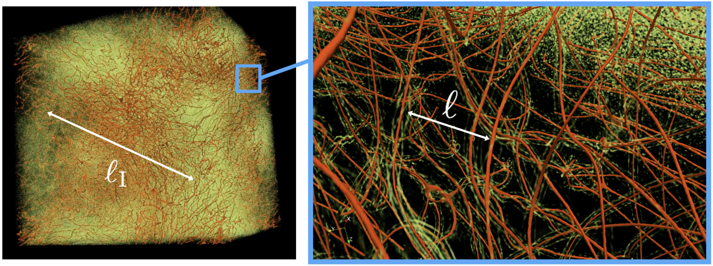
1. Vortices: Important in hydro and BEC/Optical turbulence. They may have arbitrary continuous vorticity fields in classical turbulence. In quantum turbulence, there are only infinitely thin vortices with quantised circulation. The quantum vortices are located at lines $\psi = 0$ and the circulation $\Gamma = \oint_C \mathbf{u}(\mathbf{x})\, d\ell = 2 \oint_C \nabla\theta\, d\ell = 2[\theta]_C = \pm 4\pi$.

2. Waves: Sound waves and Kelvin waves on vortex filaments are common for hydro and quantum turbulence. De Broglie (matter) waves exist in BEC/Optical systems only.

Pure vortex and pure wave turbulence are realised in the strong and weak nonlinearity limits.

Strong turbulence: vortex tangles

General vortex structure in strong 3D GP turbulence may be very complex and irregular: it is usually referred as *vortex tangle*. In a wider context, vortex tangle represents a typical realisation in superfluid turbulence at zero temperature, i.e. in liquid Helium.



N. Müller and G. Krstulovic (2020)

In what follows, we will consider the weak wave turbulence only.

Linear wave solutions

First of all, let us consider a system with weak field, such that $|\psi|^3 \ll |\nabla^2\psi|$, so that the nonlinear term in the GPE can be neglected. Then the resulting linear (Schrödinger) equation has wave solutions

$$\psi = Ae^{-i\omega_{\mathbf{k}}t + i\mathbf{k}\cdot\mathbf{x}},$$

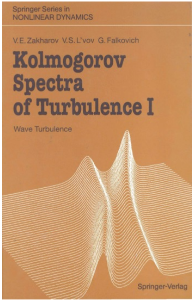
where $A =$ const is a wave amplitude, $\mathbf{k} =$ const is a wave vector, and

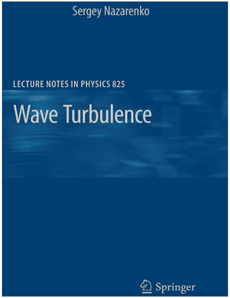
$$\omega_{\mathbf{k}} \equiv \omega(\mathbf{k}) = k^2. \tag{6}$$

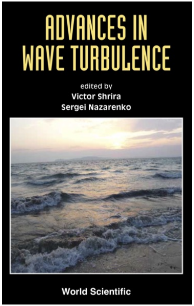
Quantum mechanical interpretation: De Broglie "matter waves".

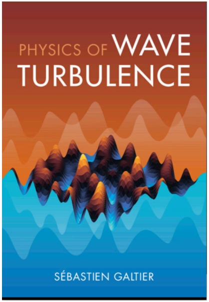
What is Wave Turbulence?

Wave Turbulence is a non-equilibrium statistical system of many randomly interacting waves. Kinetic equations of Wave Turbulence describe evolution of the wave energy in Fourier space.









Weak wave turbulence

Weak wave turbulence (WWT) refers to systems with random weakly nonlinear waves. In WWT, waveaction spectrum $n_{\mathbf{k}} = (L/2\pi)^d \langle|\psi_{\mathbf{k}}|^2\rangle$ evolves according to the wave-kinetic equation (WKE):

$$\partial_t n_{\mathbf{k}} = 4\pi \int n_{\mathbf{k}_1} n_{\mathbf{k}_2} n_{\mathbf{k}_3} n_{\mathbf{k}} \left[\frac{1}{n_{\mathbf{k}}} + \frac{1}{n_{\mathbf{k}_3}} - \frac{1}{n_{\mathbf{k}_1}} - \frac{1}{n_{\mathbf{k}_2}}\right] \times$$
$$\delta(\mathbf{k} + \mathbf{k}_3 - \mathbf{k}_1 - \mathbf{k}_2)\, \delta(\omega_{\mathbf{k}} + \omega_{\mathbf{k}_3} - \omega_{\mathbf{k}_1} - \omega_{\mathbf{k}_2})\, d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3, \tag{7}$$

where $\omega_{\mathbf{k}} = k^2$.
Now the invariants are: $N = \int n_{\mathbf{k}} d\mathbf{k}$ and $E = \int k^2 n_{\mathbf{k}} d\mathbf{k}$.

Such wave fields contain a lot of vortices (zeroes of ψ) but they are all "ghosts" without hydrodynamic properties!

Basic ideas and facts in hydrodynamic turbulence

Word "Turbulence" in Wave Turbulence is because it exhibits states similar to the ones arising in classical turbulence, energy cascades through scales. Further, the GPE system has properties similar to 2D turbulence – dual cascades. Thus, we will briefly overview the basic concepts and results in the area of classical hydrodynamic turbulence.
3D hydrodynamic turbulence, even in a statistically steady state, is far from a thermodynamic equilibrium. It is described by the energy (or anther invariant) flux through scales. It takes form as a sequence of transfers between eddies of similar sizes, and called the local energy cascade. As first suggested by Richardson, small eddies obtain the full amount of the energy contained in larger eddies when the latter are getting fragmented due to mutual interactions. Further, the small eddies are breaking into even smaller ones, and so on in a self-similar way.

The largest eddies in Richardson cascade obtain their energy from an external mechanical forcing or a large-scale instability, whereas the smallest vortices are damped by viscosity. The rate of the energy injection at the largest scales is equal, on average, to the energy dissipation rate at the smallest scales in a statistically steady state.

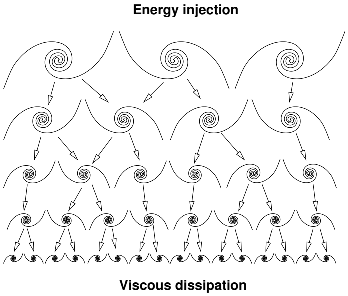


Figure: Richardson cascade in the physical space

Energy cascade in Wave Turbulence

The longest waves in the energy cascade obtain their energy from an external mechanical forcing or a large-scale instability, whereas the smallest waves are damped by viscosity (or other mechanisms). The rate of the energy injection at the largest scales is equal, on average, to the energy dissipation rate at the smallest scales in a statistically steady state.

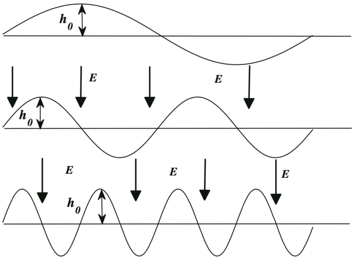


Figure: Wave energy cascade in the physical space

A **k**-space cartoon of the Richardson cascade is shown in the figure. In such a representation, the eddy size roughly corresponds to $1/k = 1/|\mathbf{k}|$. Turbulence source is at small wave numbers around k_f, and the energy cascade is in the direction of increasing k towards large dissipation wave numbers $\sim k_\nu$.

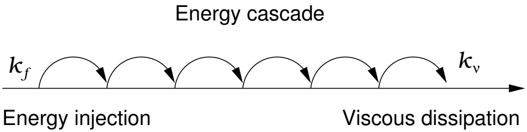


Figure: Energy cascade in the k-space

Energy Spectrum

Energy spectrum is the main statistical quantity studied in turbulence:

$$E^{(3D)}(\mathbf{k}) = \frac{1}{2} \int\limits_{\mathbb{R}^3} \langle \mathbf{u}(\mathbf{x}) \cdot \mathbf{u}(\mathbf{x}+\mathbf{r}) \rangle \, e^{-i\mathbf{k}\cdot\mathbf{r}} \frac{d\mathbf{r}}{(2\pi)^3} \ . \tag{8}$$

The angular bracket denotes a suitable average, ensemble, volume or space.

Turbulence is *homogeneous* if all averaged quantities, including the energy spectrum, are independent of $\mathbf{x}$. Respectively, in *isotropic* turbulence the energy spectrum and all the other averaged quantities are independent of the direction of the wave number $\mathbf{k}$. Thus, for the homogeneous isotropic turbulence $E^{(3D)}(\mathbf{k}) \equiv E^{(3D)}(k)$ where $k = |\mathbf{k}|$.

Super-script $(3D)$ in $E^{(3D)}$ means that it is a density in 3D **k**-space:

$$\frac{1}{2}\left\langle u^2 \right\rangle = \int\limits_{\mathbb{R}^3} E^{(3D)}(\mathbf{k})\, d\mathbf{k}. \tag{9}$$

For isotropic spectra the same information is contained in a 1D spectrum: $E^{(1D)}(k) = 4\pi k^2\, E^{(3D)}(k)$ which is the energy density over $k = |\mathbf{k}|$,

$$\frac{1}{2}\left\langle u^2 \right\rangle = \int\limits_0^{+\infty} E^{(1D)}(k) dk\ . \tag{10}$$

hence for the physical dimension: $\left[E^{(1D)}\right] = \left[\frac{\mathbf{u}^2}{k}\right] = \frac{l^3}{t^2}$

Kolmogorov and Obukhov put forward a universality hypothesis: in the so-called inertial range of scales far away from the source and the sink turbulence, $k_f \ll k \ll k_\nu$, the spectrum depends only on the energy flux through scales ϵ (defined via $\partial_t E^{(1D)} + \partial_k \epsilon = 0$), and independent from details of the forcing or the dissipation of energy and k_f and k_ν. Such an independence is natural if the Richardson cascade is local, in which the energy is transferred in many steps each involving interactions of eddies with comparable sizes only.

This leads to a dimensional argument, where ϵ and k are postulated to be the only relevant dimensional quantities in the inertial range $k_f \ll k \ll k_\nu$. The dimensions of the energy spectrum and the energy flux:

$$\left[E^{(1D)}\right] = \left[\frac{\mathbf{u}^2}{k}\right] = \frac{l^3}{t^2} \quad \text{and} \quad [\epsilon] = \left[\frac{\mathbf{u}^2}{t}\right] = \frac{l^2}{t^3} \, , \tag{11}$$

The only combination of ϵ and k that yields the correct dimension of $E^{(1D)}$ is

$$E^{(1D)} = C\epsilon^{2/3}k^{-5/3} \ . \tag{12}$$

This is famous KO41 spectrum, and C is the so-called Kolmogorov constant whose cannot be deducted from the dimensional argument alone; its experimental value is $C \sim 1.6$.

Conceptually, the Richardson cascade and the KO41 spectrum are of great importance for all turbulent systems, including the wave turbulence and the GP turbulence.

2D Turbulence

Let us consider a 2D flow in which motion in the third spatial direction is suppressed either by restraining boundaries or due to presence of a strong rotation or a magnetic field. In absence of dissipation, such a 2D flow conserves two quadratic quantities, energy and enstrophy:

$$E = \quad = \frac{1}{2}\left\langle \mathbf{u}^2 \right\rangle = \int_0^\infty E^{(1D)}(k)dk, \tag{13}$$

$$\Omega = \quad = \frac{1}{2}\left\langle \boldsymbol{\omega}^2 \right\rangle = \int_0^\infty k^2 E^{(1D)}(k)dk\ . \tag{14}$$

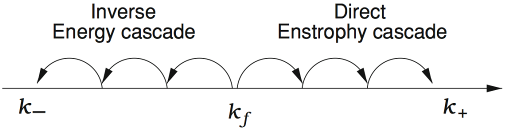
where we have taken into account $\hat{\boldsymbol{\omega}}_k = i\mathbf{k} \times \hat{\mathbf{u}}_k$.

Dual Cascade Argument (Fjørtoft'53)

Let us consider turbulence excited near a wavenumber k_f and dissipated at very small wavenumbers $k_- \ll k_f$ and at very large wavenumbers $k_+ \gg k_f$, and let there be no forcing or dissipation at wavenumbers such that $k_- < k < k_f$ or $k_f < k < k_+$, see the figure. These intervals are called the *inverse and the direct cascade inertial ranges* respectively. From (13) and (14):

$$\eta_+ \sim k_+^2 \epsilon_+, \quad \eta_- \sim k_-^2 \epsilon_-, \quad \eta \sim k_f^2 \epsilon,$$
$$\eta_+ + \eta_- = \eta, \quad \epsilon_+ + \epsilon_- = \epsilon. \tag{15}$$

So, $\epsilon_+/\epsilon = (k_f^2 - k_-^2)/(k_+^2 + k_-^2) \to 0$ and $\epsilon_-/\epsilon \to 1$. Similarly $\eta_+/\eta \to 0$ and $\eta_-/\eta \to 1$.



Spectra of 2D turbulence

Following the ideas of Kolmogorov and Obukhov, Kraichnan assumed that the energy spectrum in the inverse and the direct cascade ranges is determined by only the energy or the enstrophy flux respectively.
By a dimensional argument, he obtained for the inverse cascade range:

$$E^{(1D)}(k) = C_\epsilon \epsilon^{2/3} k^{-5/3} \ .$$

The constant cannot be obtained dimensionally; numerics yield $C_\epsilon \sim 6$.
To find the enstrophy cascade spectrum, we first find the dimension for η:

$$[\eta] = \left[k^2\right] [\epsilon] = \frac{1}{l^2} \frac{l^2}{t^3} = \frac{1}{t^3} \ . \tag{16}$$

Spectra of 2D turbulence cont'd

Therefore, in the direct cascade range, assuming that the spectrum can only depend on η and k, we have:

$$E^{(1D)} = C_\eta \eta^{2/3} k^{-3} \, , \tag{17}$$

which is Kraichnan's spectrum. Here $C_\eta \sim 1.9$ — a value obtained by numerics.

Conservation laws and cascade directions in WT

Dual cascades of 2D turbulence are of direct relevance to the GP turbulence: like in 2D turbulence, there are two positive GP invariants, E and N, which, as we will show now, cascade in the opposite directions. The kinetic equation conserves the total energy:

$$\dot{E} = \frac{d}{dt} \int \omega_{\mathbf{k}} n_{\mathbf{k}} \, d\mathbf{k} = 0. \tag{18}$$

and the total number of particles (waveaction):

$$\dot{N} = \frac{d}{dt} \int n_{\mathbf{k}} \, d\mathbf{k} = 0. \tag{19}$$

For GPE, $\omega_{\mathbf{k}} = k^2$. So for Fjørtoft's argument, we have a mapping from 2D turbulence to GPE: $E \to N, \Omega \to E$.

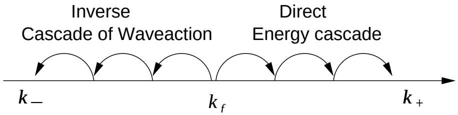


Figure: Dual cascade in weak wave turbulence

Dual cascade in BEC

In BEC, the dual behavior has a nice physical interpretation. Consider a system in a trap, as shown in the figure. In this setup, the forward cascade will correspond to an energy transfer toward larger energy levels. When such an energy cascade reaches highest available levels in the trap, it will "spill" over the potential barrier. This corresponds to *evaporative cooling*, a technique used experimentally in BEC experiments.

Yu. Lvov et al./Physica D 184 (2003) 333–351

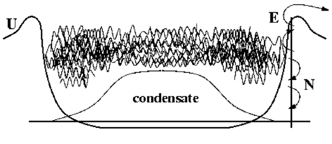
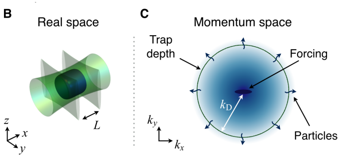


Fig. 1. Turbulent cascades of energy E and particle number N.

BEC Turbulence experiment of Navon et al.'2018.



- Direct E-cascade: "evaporation".
- Inverse N-cascade: Non-equilibrium condensation.

Kolmogorov-Zakharov spectra

KZ spectra are direct analogs of the Kolmogorov-Obukhov spectrum, and can also be obtained dimensionally, but (provided they satisfy the locality test) the KZ spectra are also exact power law solutions

$$n_{\mathbf{k}} = Ak^{\nu}$$

of the wave-kinetic equation. First, we will obtain the KZ spectra by considering physical dimensions. A disadvantage of such an approach is that we will not be able to check locality of the KZ spectra manifested in convergence of the integrals in the wave-kinetic equations.

Kolmogorov-Zakharov spectra

Write the kinetic equation in the form of the energy balance equation:

$$\dot{E}_k^{(1D)} = S\, k^{d-1} \omega_{\mathbf{k}} \dot{n}_{\mathbf{k}} = -\partial_k P, \tag{20}$$

where S is a constant equal to the area of the unit sphere in the d-dimensional space and $\omega_{\mathbf{k}} = k^2$.

$$\dot{n}_{\mathbf{k}} = \int n_{\mathbf{k}_1} n_{\mathbf{k}_2} n_{\mathbf{k}_3} n_{\mathbf{k}} \left[\frac{1}{n_{\mathbf{k}}} + \frac{1}{n_{\mathbf{k}_3}} - \frac{1}{n_{\mathbf{k}_1}} - \frac{1}{n_{\mathbf{k}_2}} \right] \delta(\mathbf{k})\, \delta(\omega_{\mathbf{k}})\, d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3,$$

Let $n_{\mathbf{k}} \sim k^{\nu}$. From (20) we have $P \sim k^{3d+3\nu}$, so for the constant flux state $P \sim k^0$, which results in:

$$\nu = \nu_E = -d \qquad \text{(direct cascade)}. \tag{21}$$

Kolmogorov-Zakharov spectra

One can apply the same dimensional analysis to the the inverse waveaction cascade. For that one needs to write the wave-kinetic equation in the waveaction conservation form in terms of the waveaction flux Q:

$$\dot{n}_k^{(1D)} \equiv S\, k^{d-1}\, \dot{n}_{\mathbf{k}} = -\partial_k Q. \tag{22}$$

$Q \sim k^{3d+3\nu-2}$, so for the constant flux state, $Q \sim k^0$, we have the following exponent of the spectrum:

$$\nu = \nu_N = -d + 2/3 \qquad \text{(inverse cascade)}. \tag{23}$$

Beyond dimensional analysis: rigorous results and numerics

Zhu et al, PRL 2023

Beyond dimensional analysis: rigorous results and numerics

KZ spectra are only meaningful if they are *local*, i.e. when the integral defining the flux (P or Q) converges. Let us now derive the KZ spectra rigorously, check locality and find the KZ constants.
Kinetic equation in frequency variable:

$$\frac{\partial n_\omega}{\partial t} = \frac{4\pi^3}{\sqrt{\omega}} \int \left[\min\left(\omega, \omega_1, \omega_2, \omega_3\right)\right]^{1/2} n_\omega n_{\omega_1} n_{\omega_2} n_{\omega_3} \left(\frac{1}{n_\omega} + \frac{1}{n_{\omega_1}} - \frac{1}{n_{\omega_2}} - \frac{1}{n_{\omega_3}}\right) \delta(\omega_{23}^{01}) d\omega_1 d\omega_2 d\omega_3, \qquad (24)$$

where now $\omega_{23}^{01} = \omega + \omega_1 - \omega_2 - \omega_3$. N and E in terms of ω:

$$N = 2\pi \int_0^\infty \omega^{1/2} n(\omega, t) d\omega\,, \qquad E = 2\pi \int_0^\infty \omega^{3/2} n(\omega_1, t) d\omega\,. \qquad (25a-b)$$

Fluxes and power law spectra

N and E balance equations:

$$\partial_t(\omega^{1/2} n(\omega,t)) + \partial_\omega Q = 0, \qquad \partial_t(\omega^{3/2} n(\omega_1,t)) + \partial_\omega P = 0\,. \tag{26a-b}$$

where N and E fluxes are

$$Q(\omega,t) = -2\pi \int_0^\omega \omega_1^{1/2} St(\omega_1,t) d\omega_1\,, \quad P(\omega,t) = -2\pi \int_0^\omega \omega_1^{3/2} St(\omega_1,t) d\omega_1 \tag{27a-b}$$

Substitute $n_\omega = A\omega^{-x}$ (not necessarily a WKE solution) into the WKE:

$$\frac{\partial n_\omega}{\partial t} = 4\pi^3 A^3 \omega^{-3x+2} I(x)\,, \tag{28}$$

with

$$I(x) = \int_{q_1,q_2,q_3>0} [\min(1,q_1,q_2,q_3)]^{1/2} (q_1 q_2 q_3)^{-x} (1 + q_1^x - q_2^x - q_3^x)\, \delta\left(q_{23}^{01}\right) dq_1 dq_2 dq_3\,, \tag{29}$$

$q_i = \omega_i/\omega$ for $i = 1,2,3$.

Zakharov transformation (ZT)

If we integrate over q_1, the integration domain in the (q_2, q_3) plane is over: $q_2, q_3 > 0\,, q_2 + q_3 - 1 = q_1 > 0$. Apply ZT:

$$
\begin{aligned}
q_2 &= \frac{1}{\tilde{q}_2}\,, \quad q_1 = \frac{\tilde{q}_3}{\tilde{q}_2}\,, \quad q_3 = \frac{\tilde{q}_1}{\tilde{q}_2}\,, \quad \text{for} \quad q_2 > 1\,, 0 < q_3 < 1\,,\\
q_3 &= \frac{1}{\tilde{q}_3}\,, \quad q_1 = \frac{\tilde{q}_2}{\tilde{q}_3}\,, \quad q_2 = \frac{\tilde{q}_1}{\tilde{q}_3}\,, \quad \text{for} \quad 0 < q_2 < 1\,, q_3 > 1\,,\\
q_1 &= \frac{1}{\tilde{q}_1}\,, \quad q_2 = \frac{\tilde{q}_3}{\tilde{q}_1}\,, \quad q_3 = \frac{\tilde{q}_2}{\tilde{q}_1}\,, \quad \text{for} \quad q_2\,, q_3 > 1\,.
\end{aligned}
\tag{30}
$$

After dropping tildes, $I(x)$ becomes

$$
I_{\mathrm{ZT}}(x) = \int q_1^{1/2-x}(q_2 q_3)^{-x}(1 + q_1^x - q_2^x - q_3^x)(1 + q_1^y - q_2^y - q_3^y)\,\delta(q_{23}^{01})dq_1\,dq_2\,dq_3, \tag{31}
$$

with $y = 3x - 7/2$, and the integration now is over $0 < q_1\,, q_2\,, q_3 < 1$.

Note: $I(x) = I_{\mathrm{ZT}}(x)$ only if the integrals are convergent.

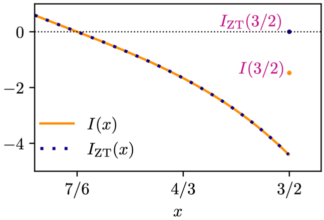


Figure: $I(x)$ and $I_{\mathrm{ZT}}(x)$ in the window of convergence $1 < x < 3/2$ of $I(x)$.

$I_{\mathrm{ZT}}(x)$ has two zeros: $x = 3/2$ ($y = 1$) corresponding to the forward cascade of energy $n_\omega = A\omega^{-3/2}$ and $x = 7/6$ ($y = 0$)—to the inverse cascade of particles $n_\omega = A\omega^{-7/6}$. However, the ZT is not an identity transformation and, therefore, the candidates to the stationary solutions must be checked by substituting them into the original integral $I(x)$ and making sure that it is convergent and equal to zero. Physically, the integral convergence means that the wave quartets with similar values of the frequencies dominate the nonlinear evolution; this is the so-called interaction locality. Mathematically, violation of locality simply means that the considered spectrum is not a valid solution.

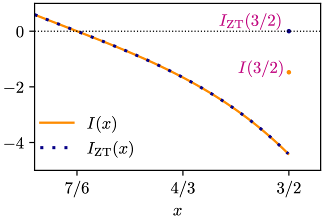


Figure: $I(x)$ and $I_{\mathrm{ZT}}(x)$ in the window of convergence $1 < x < 3/2$ of $I(x)$.

$I(x)$ and $I_{\mathrm{ZT}}(x)$ do coincide in interval $1 < x < 3/2$. Therefore, the inverse cascade spectrum $n_\omega = A\omega^{-7/6}$ is local, and a valid mathematical solution of the WKE. Further, $I(3/2)$ is convergent and, using Mathematica, $I(3/2) = -4\pi + 16 \ln 2$. Fact $I(3/2) \neq 0$ implies that, although the collision integral of the WKE is convergent for $x = 3/2$, the spectrum with this exponent is not an exact stationary solution of the WKE. However, with a logarithmic correction this spectrum can be made a valid asymptotical solution for the direct cascade.

Plug spectrum $n = Ak^{-x}$ into the N-balance equation:

$$\frac{\partial \left(2\pi\omega^{1/2} n_\omega\right)}{\partial t} = -\frac{\partial Q(\omega, t)}{\partial \omega} \equiv 8\pi^4 A^3 \omega^{-3x+5/2} I_{\mathrm{ZT}}(x).$$

$$\text{with } Q(\omega, t) = \int_0^\omega \left(-8\pi^4 A^3 \omega_0^{-3x+5/2} I_{\mathrm{ZT}}(x)\right) d\omega_0 = 8\pi^4 A^3 \omega^{-y} \frac{I_{\mathrm{ZT}}(x)}{3x - 7/2}.$$

For $x \to 7/6$ ($y \to 0$), by L'Hopital $Q = 8\pi^4 A^3 I'_{ZT}(7/6)/3$, so

$$n_k = n_\omega = 3^{1/3} \left(8\pi^4 I'_{ZT}(7/6)\right)^{-1/3} Q^{1/3} \omega^{-7/6}.$$

Using Mathematica, we find:

$$C_{\mathrm{i}} = \frac{1}{2\pi^{3/2}} \Gamma\left(\tfrac{5}{6}\right)^{1/3} \left[3\Gamma\left(\tfrac{1}{3}\right) \left(3^{3/2} 2^{2/3} {}_3F_2\left(\tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{3}; \tfrac{4}{3}, \tfrac{4}{3}; 1\right) - 8\, {}_3F_2\left(\tfrac{1}{6}, \tfrac{1}{3}, \tfrac{1}{3}; \tfrac{4}{3}, \tfrac{3}{2}; 1\right)\right.\right.$$
$$\left.\left. + 2^{1/3} {}_3F_2\left(\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{2}; \tfrac{3}{2}, \tfrac{5}{3}; 1\right) - 2^{1/3} {}_4F_3\left(\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{2}, \tfrac{1}{2}; \tfrac{3}{2}, \tfrac{3}{2}, \tfrac{5}{3}; 1\right)\right)\right]^{-1/3} \approx 7.5774045 \times 10^{-2}.$$

Now, plug spectrum $n = Ak^{-x}$ into the E-balance equation:

$$\frac{\partial\left(2\pi\omega^{3/2}n_\omega\right)}{\partial t} = -\frac{\partial P(\omega,t)}{\partial\omega} \equiv 8\pi^4 A^3\,\omega^{-3x+7/2}\,I(x)\,, \qquad (32)$$

$$P(\omega,t) = \int_0^\omega \left(-8\pi^4 A^3\,\omega_0^{-3x+7/2}\,I(x)\right) d\omega_0\,. \qquad (33)$$

Since $0 < I(3/2) = \text{const} < \infty$, Eq. (33) gives a logarithmically divergent integral for $x = 3/2$. Cutting off at the forcing frequency $\omega_\mathrm{f} = k_\mathrm{f}^2$:

$$P = \int_{\omega_\mathrm{f}}^\omega \left(-8\pi^4 A^3\,\omega_0^{-1}\,I(3/2)\right) d\omega_0 = -8\pi^4 A^3\,I(3/2)\ln\frac{\omega}{\omega_\mathrm{f}}\,. \qquad (34)$$

But flux P must be independent of ω for steady state solutions. This is clearly not the case in the above expression, which is another indication that $\sim\omega^{-3/2}$ is not a valid stationary solution of the WKE.
To remove the ω-dependence term $\ln\frac{\omega}{\omega_\mathrm{f}}$ in P we introduce a logarithmic correction and seek a solution as $n_\omega = C\omega^{-x}\ln^z\frac{\omega}{\omega_\mathrm{f}}$.

Thus, we obtain

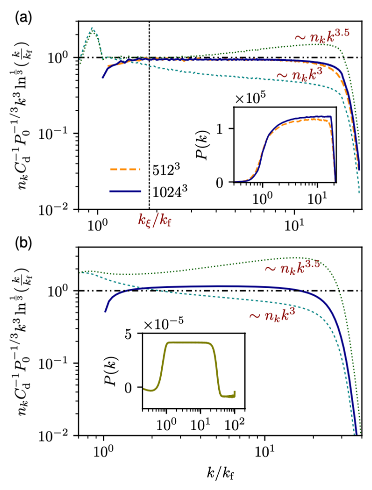
$$P = -\,8\pi^4 C^3\, I(x)\ln^{3z}\frac{\omega}{\omega_{\rm f}}\int_{\omega_{\rm f}}^{\omega}\omega_0^{-3x+7/2}d\omega_0\,. \tag{35}$$

Independence of P from ω requires $x=-3/2$ and $z=-1/3$,
$n_\omega = \left(-8\pi^4 I(3/2)\right)^{-1/3} P^{1/3}\omega^{-3/2}\ln^{-1/3}(\omega/\omega_{\rm f})$, then

$$n_k = \left(-16\pi^4 I(3/2)\right)^{-1/3} P^{1/3} k^{-3}\ln^{-1/3}(k/k_{\rm f})\,. \tag{36}$$

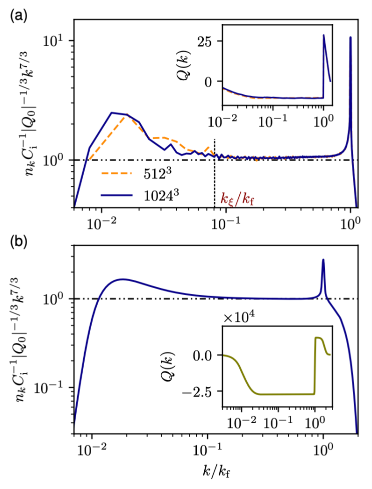
The analytical expression $I(3/2) = -4\pi + 16\ln 2$ gives $C_{\rm d}\approx 7.58\times 10^{-2}$.
Numerically obtained $I(3/2^-)\approx -4.42$ gives an alternative $C_d \approx 5.26\times 10^{-2}$.

Compensated wave action spectra by theoretical predictions



- **(a)** Simulations of GPE at two different resolutions
- **(b)** Simulation of WKE.
- Insets: energy fluxes.
- Horizontal lines—perfect theoretical predictions.
- $\sim k^{-3.5}$ fits better than $\sim k^{-3}$.
- Best agreement is the log-corrected KZ spectrum. Bump of the uncompensated spectrum at k_f – signature of the singularity of $\log^{-1/3}(k/k_f)$ at k_f.

Compensated wave action spectra by theoretical predictions



- **(a)** Simulations of GPE at two different resolutions
- **(b)** Simulation of WKE.
- Insets: particle fluxes.
- Almost perfect agreement for WKE data.
- Around 5% deviation for GPE data.

Summary

- Nonlinear chaotic motions of waves in BEC have properties of turbulence: dual cascades, Kolmogorov-like scalings.
- Inverse cascade of the waveaction/particles is a process of nonequilibrium condensation. Direct cascade of energy is an "evaporative cooling".
- Direct cascade in experimental BEC WT is explained. Inverse cascade is awaiting experimental implementation.

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