On the role of mutual friction in He-II Turbulence

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with

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Outline:

PART A: brief introduction on superfluid He\(^4\) (called He-II)

- Fluid-Mechanical approach of He-II dynamics ; two-fluid approach
- Coarse-grained dynamics
- HVBK closure conditions

PART B: Quantum Turbulence under HVBK closure conditions

- Scale-by-scale energy cascade budget of He-II mixture
- Introduction of effective characteristic scales and Reynolds numbers
- Intermittency and anomalous scaling laws ; temperature dependence
PART A

Brief introduction on superfluid He⁴ (called He-II)
Superfluid He\textsuperscript{4}—general features

Superfluid He\textsuperscript{4} is an archetype of quantum fluid

A superfluid is not a « perfect » Euler fluid with kinematic viscosity $\nu \to 0$ but behaves as if it consists of two components (co-penetrating fluids):

- **inviscid superfluid component** (quantum ground state: vortex line)

- **normal fluid component** (excited states: phonons and rotons)

\[ \rho = \rho_n + \rho_s \]
\[ \rho u = \rho_n u^n + \rho_s u^s \]
Length scales in the problem

$d = \text{vortex core length scale}$

$\ell = \text{intervortex distance} \ll dx = \text{grid resolution}$

$Kivotides, 2006$

$dx = \text{grid resolution} \gg \ell = \text{intervortex distance}$
Coarse-grained dynamics of He-II at $1K < T < T_{\lambda} = 2.17K$

- superfluid component:
- normal-fluid component:

superfluid component:

normal-fluid component:

course-graining
$\Delta x \gg$ intervortex distance

Vortex tangle

Euler fluid

Navier-Stokes fluid with viscosity $\nu_n$

with mutual coupling

microscopic = semi-classical approach, e.g. Gross Pitaevskii

macroscopic

+ subgrid scale dissipation: vortex reconnection, etc.
• He-II is considered isothermal and incompressible

\[ \rho_n \frac{D u^n}{Dt} = -\nabla p_n + F_{hvbk} + \nu_n \nabla^2 u^n \]

\[ \rho_s \frac{D u^s}{Dt} = -\nabla p_s - F_{hvbk} + "\text{subgrid dissipation}" \]

\( \omega^s = \nabla \times u^s \)  
**superfluid vorticity** accounts for the underlying tangle of quantized vortices

\[ F_{hvbk} \left( \frac{\rho_n \rho_s}{\rho}, \omega^s, u^n - u^s, B, B' \right) \approx -\frac{B}{2} \frac{\rho_n \rho_s}{\rho} \frac{\omega^s}{|\omega^s|} \times (\omega^s \times (u^s - u^n)) + \frac{B'}{2} \frac{\rho_n \rho_s}{\rho} \omega^s \times (u^s - u^n) \approx -\frac{B}{2} \frac{\rho_n \rho_s}{\rho} |\omega^s| (u^n - u^s) \]

most relevant for Quantum Turbulence
Limitations and alternatives to HVBK closure condition

HVBK was originally designed for regular pattern of quantized vortices with parallel orientation

\[ F_{\text{hvbk}} \approx -\frac{B}{2} \frac{\rho_n \rho_s}{\rho} \kappa L(u^n - u^s) \]

\[ L \approx \frac{|\omega^s|}{\kappa} \quad \text{vortex line density} \]

In the context of QT, it is expected to capture only the locally polarized contribution of the superfluid vortex tangle: \(|\omega^s| = \kappa L_\parallel\)

Randomly oriented quantized vortices also participate to mutual friction: \(\kappa L = \kappa (L_\parallel + L_\perp)\) but are not taken into account

Vinen's model accounts for locally non-polarized superfluid vortex tangle:

\[ F_{\text{Vinen}} = -\alpha(T) \rho_s \kappa L_\chi (u^n - u^s) \]

\[ \frac{dL_\chi}{dt} = \alpha_V |u^n - u^s| \frac{3}{L_\chi^2} - \beta_V L_\chi^2 \]

production \quad \text{decay}

Attempts to unify Vinen's model and HVBK closure condition (T. Lipniacki):

\[ q = \frac{\nabla \times u^s}{\kappa L} \quad \text{measure of local anisotropy of the superfluid vortex tangle} \]

\[ q = 1 : \text{polarized vortices} \]

\[ q = 0 : \text{isotropic, unpolarized tangle} \]
PART B
Quantum Turbulence under HVBK closure conditions
Mainstream consensus

Quantum Turbulence is **probably** very similar to Classical Turbulence

To what extent: scale-by-scale comparison; dependence on temperature?
Coarse-grained dynamics of QT under HVBK closure conditions
 Mathematical framework of our study

No temperature effect
Momentum equation for normal fluid and superfluid components:

\[ \frac{\partial \mathbf{u}^n}{\partial t} + (\mathbf{u}^n \cdot \nabla)\mathbf{u}^n = -\frac{1}{\rho_n} \nabla p_n + \frac{\rho_s}{\rho} \mathbf{F}^{ns} + \nu_n \nabla^2 \mathbf{u}^n \quad \nabla \cdot \mathbf{u}^n = 0 \]

\[ \frac{\partial \mathbf{u}^s}{\partial t} + (\mathbf{u}^s \cdot \nabla)\mathbf{u}^s = -\frac{1}{\rho_s} \nabla p_s - \frac{\rho_n}{\rho} \mathbf{F}^{ns} + \nu_s \nabla^2 \mathbf{u}^s \quad \nabla \cdot \mathbf{u}^s = 0 \]

to account for dissipation beyond inter-vortex scale: reconnection, Kelvin waves, sound emission, etc.

\[ \frac{\rho_n \rho_s}{\rho^2} \mathbf{F}^{ns} \]
is the mutual coupling force per unit mass of He-II with

\[ \mathbf{F}^{ns} = -\frac{B}{2} |\mathbf{\omega}^s| (\mathbf{u}^n - \mathbf{u}^s) \]

for the HVBK closure condition; \( \mathbf{\omega}^s = \nabla \times \mathbf{u}^s \) is the coarse grained superfluid vorticity

\[ \mathbf{F}^{ns} = -\frac{B}{2} \kappa L (\mathbf{u}^n - \mathbf{u}^s) \]

in a mean-field approximation with \( L \) being the (uniform) vortex line density
Wavenumber-by-wavenumber energy budget of QT

\[ \frac{\rho_s}{\rho_n} = 1 \text{ at } T = 1.96 \text{K} \]

FIG. 1. Scale-by-scale spectral energy fluxes in (a) the superfluid component, (b) the normal fluid component, and (c) the two-fluid mixture. Blue solid lines, energy flux \( \Pi(k) \) through wave number \( k \); orange dashed lines, viscous dissipation \( \mathcal{D}(k) \); green dotted lines, energy transfers due to mutual friction. Results obtained from HVBK simulations at \( T = 1.96 \text{K} \) (case \( \triangledown \) in Table 1).
$\frac{\nu_n}{\nu_s} = 5$

$\frac{\rho_s}{\rho_n} = 0.1$

$\frac{\rho_s}{\rho_n} = 1$

$\frac{\rho_s}{\rho_n} = 10$
Scale-by-scale energy budget in classical turbulence

Under the assumption of **stationary homogeneous and isotropic turbulence**, an exact scale-by-scale energy budget can be derived from the Navier-Stokes equations,

\[ \epsilon = \epsilon_{\text{inj}} \]

\( \epsilon \) refers to the mean dissipation rate (per unit mass)

\[ < |\delta u_i|^2(r) \delta u_\parallel(r) > = -\frac{4}{3} \epsilon_{\text{inj}} r + 2\nu \frac{d < |\delta u_i|^2(r) >}{dr} \]

for \( r \ll L_0 \): energy injection scale

This eventually yields the famous **Kolmogorov’s four-fifth law**

\[ < \delta u_\parallel^3(r) > = -\frac{4}{5} \epsilon_{\text{inj}} r + 6 \nu \frac{d < \delta u_\parallel^2(r) >}{dr} \]
Scale-by-scale energy budget in Quantum Turbulence for normal fluid and superfluid components individually

Normal fluid:
\[
\epsilon_{\text{inj}}^n = \epsilon^n - \frac{\rho_s}{\rho} < u_i^n F_i^{ns} >
\]

\[
< |\delta u_i^n|^2(r) \delta u_{||}^n(r) > = -\frac{4}{3} \epsilon_{\text{inj}}^n r - \frac{\rho_s}{\rho} \frac{2}{r^2} \int_0^r \left( < u_i^n F_i^{ns} > + < u_i^n F_i^{ns'} > \right) r'^2 dr' + 2 \nu_n \frac{d}{dr} < |\delta u_i^n|^2(r) >
\]

for \( r \ll L_0 \): energy injection scale

Superfluid:
\[
\epsilon_{\text{inj}}^s = \epsilon^s + \frac{\rho_n}{\rho} < u_i^s F_i^{ns} >
\]

\[
< |\delta u_i^s|^2(r) \delta u_{||}^s(r) > = -\frac{4}{3} \epsilon_{\text{inj}}^s r + \frac{\rho_n}{\rho} \frac{2}{r^2} \int_0^r \left( < u_i^s F_i^{ns} > + < u_i^s F_i^{ns'} > \right) r'^2 dr' + 2 \nu_s \frac{d}{dr} < |\delta u_i^s|^2(r) >
\]

for \( r \ll L_0 \): energy injection scale
Scale-by-scale energy budget in Quantum Turbulence for the two-fluid He-II mixture

\[
\frac{\rho_n}{\rho} < |\delta u_i^n|^2 \delta u_i^n > + \frac{\rho_s}{\rho} < |\delta u_i^s|^2 \delta u_i^s > = \\
- \frac{4}{3} \left( \frac{\rho_n}{\rho} \epsilon_{\text{inj}}^n + \frac{\rho_s}{\rho} \epsilon_{\text{inj}}^s \right) r \\
+ 2 \left( \frac{\rho_n}{\rho} \nu_n \frac{d}{dr} < |\delta u_i^n|^2 > + \frac{\rho_s}{\rho} \nu_s \frac{d}{dr} < |\delta u_i^s|^2 > \right) \\
- \frac{\rho_s \rho_n}{\rho^2} \frac{2}{r^2} \int_0^r < u_i^{ns'} F_i^{ns'} + u_i^{ns} F_i^{ns'} > r'^2 dr'
\]

with \( u_i^{ns} \equiv u_i^n - u_i^s \)

Supplementary term compared to Classical Turbulence
At which scale is it effective?

\[
\bar{S}_3(r) = - \frac{4}{3} \bar{\epsilon}_{\text{inj}} r + 2 \bar{\nu} \frac{d}{dr} \bar{S}_2(r) + \varphi_{ns}(r)
\]

mass-density weighted scale-by-scale energy budget of He-II
\[
\varphi_{ns}(r) = -\frac{\rho_s \rho_n}{\rho^2} \frac{2}{r^2} \int_0^r < u_i^{ns'} F_i^{ns} + u_i^{ns} F_i^{ns'} > r'^2 dr'
\]

\[
\epsilon_{ns} = \frac{\rho_s \rho_n}{\rho^2} < u_i^{ns} F_i^{ns} >
\]

If HVBK: \[
F_i^{ns} = -\frac{B}{2} |\omega^s| u_i^{ns} \quad \text{with} \quad |\omega^s| \approx \sqrt{<|\omega^s|^2>} = \kappa L
\]

\[
\varphi_{ns}(r) = -\frac{4\epsilon_{ns}}{r^2} \int_0^r \frac{< u_i^{ns'} F_i^{ns} + u_i^{ns} F_i^{ns'} >}{2 < u_i^{ns} F_i^{ns} >} r'^2 dr'
\]

\[
\varphi_{ns}(r) = 2\epsilon_{ns} \frac{2}{r^2} \int_0^r \frac{< u_i^{ns'} u_i^{ns} >}{< |u^{ns}|^2 >} r'^2 dr'
\]

**Mutual coupling – energy budget**

mean-field approximation
Mutual coupling – scale behaviour

\[ \phi_{ns}(r) = 2\epsilon_{ns} \frac{2}{r^2} \int_0^r \frac{<u_i^{ns'}u_i^{ns}>}{<|u^{ns}|^2>} r'^2 dr' \]

As \( r \to 0 \): \( \phi_{ns}(r) \to \frac{4}{3} \epsilon_{ns} r \)

\( \frac{4}{3} \epsilon_{ns} r \) \( (\epsilon_{ns} < \epsilon_{inj}) \)

\( \phi_{ns}(r) \) negligible compared to \( \frac{4}{3} \epsilon_{inj} r \)

\textit{frictional dissipation is negligible compared to energy cascade flux}

frictional dissipation is negligible compared to energy cascade flux

\( \phi_{ns}(r) \) at large scales \( \sim \frac{1}{r^2} \)
Effective Kolmogorov’s scale for He-II

In classical turbulence: Kolmogorov’s scale is obtained by equaling

- \( S_2(r) \sim < \left( \frac{\partial u_x}{\partial x} \right)^2 > r^2 \sim \frac{\epsilon}{\nu} r^2 \) in the “dissipative range” where velocity field is regular (smooth)
- and \( S_2(r) \sim \epsilon^{2/3} r^{2/3} \) in the “inertial range” where velocity field is irregular

This yields \( \eta \sim \left( \frac{\nu^3}{\epsilon} \right)^{1/4} \)

Analogously in quantum turbulence: effective Kolmogorov’s scale would be obtained by equaling

- \( \overline{S_2}(r) \sim \frac{< \rho_n \nu_n \left( \frac{\partial u_x^n}{\partial x} \right)^2 + \rho_s \nu_s \left( \frac{\partial u_x^s}{\partial x} \right)^2 >}{\rho_n \nu_n + \rho_s \nu_s} r^2 \sim \frac{\bar{\epsilon}_v}{\bar{\nu}} r^2 \) in the “dissipative range” where velocity field is regular
- and \( \overline{S_2}(r) \sim \bar{\epsilon}_{inj}^{2/3} r^{2/3} \) in the “inertial range” where velocity field is regular

This yields

\[
\overline{\eta} \sim \left( \frac{\bar{\nu}^3}{\bar{\epsilon}_{inj}} \right)^{1/4} \times \left( \frac{\bar{\epsilon}_{inj}}{\bar{\epsilon}_{inj} - \epsilon_{ns}} \right)^{3/4} \text{ by considering } \bar{\epsilon}_{inj} = \bar{\epsilon}_v + \epsilon_{ns}
\]

Or equivalently:

\[
\overline{\eta} \sim \left( \frac{\nu_{eff}^3}{\bar{\epsilon}_{inj}} \right)^{1/4} \text{ with } \nu_{eff} = \bar{\nu} \times \left( \frac{\bar{\epsilon}_{inj}}{\bar{\epsilon}_{inj} - \epsilon_{ns}} \right)
\]

Enhanced viscosity that accounts for mutual friction
Effective Reynolds number for He-II

Effective Reynolds number:

\[
Re = \frac{u_{\text{rms}}L_0}{\bar{v}_{\text{eff}}} \sim \left(\frac{L_0}{\bar{\eta}}\right)^{\frac{3}{4}}
\]

\[
\bar{\eta} \sim \left(\frac{\bar{v}_{\text{eff}}}{\bar{c}_{\text{inj}}}\right)^{\frac{1}{4}} \text{ with } \bar{v}_{\text{eff}} = \bar{v} \left(\frac{\bar{c}_{\text{inj}}}{\bar{c}_{\text{inj}} - \epsilon_{\text{ns}}}\right) = \bar{v} \left(1 + \frac{\epsilon_{\text{ns}}}{\epsilon_{\nu}}\right)
\]

Effective Taylor microscale \(\bar{\lambda}\) is defined by

\[
\bar{c}_{\text{inj}} = 15 \bar{v}_{\text{eff}} \left(\frac{u_{\text{rms}}}{\bar{\lambda}}\right)^2
\]

or equivalently

\[
\bar{c}_{\nu} = 15\bar{v} \left(\frac{u_{\text{rms}}}{\bar{\lambda}}\right)^2
\]

\[
R_{\bar{\lambda}} = \frac{u_{\text{rms}}\bar{\lambda}}{\bar{v}_{\text{eff}}} \sim \sqrt{Re}
\]
Mutual friction for He-II mixture is essentially a dissipative process that adds to viscous dissipation at very small scales.
Experimental studies:

- "Local investigation of superfluid turbulence"
  J. Maurer and P. Tabeling

- "Intermittency of QT with superfluid fraction from 0% to 96%"
  E. Rusaouen, B. Chabaud, J. Salort, P-E Roche

- "... No evidence of temperature dependence is found on these scaling exponents in the upper part of the inertial cascade, where turbulence is well developed and fully resolved by the probe..."

- "Intermittency enhancement in quantum turbulence in superfluid 4He"
  Emil Varga, Jian Gao, Wei Guo, and Ladislav Skrbek

- "... measurements reveal temperature-dependent intermittency corrections (on transverse velocity structure functions) that peak in the vicinity of 1.85K in excellent agreement with recent theoretical predictions ..."
Controversy on intermittency in Quantum Turbulence

Numerical results

Numerical studies:

- "Enhancement of Intermittency in Superfluid Turbulence”
  Laurent Boué, Victor L’vov, Anna Pomyalov, and Itamar Procaccia

- "Multiscaling in superfluid turbulence: A shellmodel study”
  V. Shukla and R. Pandit

- "Turbulent statistics and intermittency enhancement in coflowing superfluid 4He”
  L. Biferale, D. Khomenko, V. L’vov, A. Pomyalov, I. Procaccia, and G. Sahoo

"...The energy transfer by mutual friction between components is particularly efficient in the temperature range between 1.8 and 2 K, leading to enhancement of small-scale intermittency for these temperatures..."
Flatness of longitudinal increments for normal fluid and superfluid components

Inset = results from Biferale et al.

Enhancement of intermittency due to dissipative effects depends on temperature through the mutual friction

Universal scaling exponent for all temperatures similar to classical turbulence

TABLE I. Simulation parameters.

<table>
<thead>
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<th>Key</th>
<th>$T$ (K)</th>
<th>$\rho_f/\rho_n$</th>
<th>$v_s/v_m$</th>
<th>$B$</th>
<th>$N$</th>
<th>$k_{max}/\overline{\xi}$</th>
<th>$\Lambda/\overline{\eta}$</th>
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<td>19.6</td>
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</tbody>
</table>
Mutual friction for He-II mixture is essentially a dissipative process that adds to viscous dissipation at very small scales.

This allows us to define an effective Kolmogorov scale

$$\eta \sim \left( \frac{\varepsilon_{\text{eff}}}{\varepsilon_{\text{inj}}} \right)^{\frac{1}{4}}$$

with

$$\nu_{\text{eff}} = \nu \times \left( \frac{\varepsilon_{\text{inj}}}{\varepsilon_{\text{inj}} - \varepsilon_{\text{ns}}} \right)$$

and an effective Reynolds number

$$Re = \frac{u_{\text{rms}}L_0}{\nu_{\text{eff}}} \sim \left( \frac{L_0}{\eta} \right)^{\frac{3}{4}}$$
Conclusion – Take Home messages

- In the inertial range, the energy cascade remains the dominant dynamical process:
  - the two fluid components are locked
  - scaling properties agree with that of classical turbulence for all temperatures

Fully consistent with Zhang Z, Danaila I, Lévêque E, Danaila L. ``Higher-order statistics and intermittency of a two-fluid Hall-Vinen-Bekharevich-Khalatnikov quantum turbulent flow” J. Fluid Mech.. 2023;962:A22