Time arrow in reconnecting dynamics of vortices

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高知工科大学
KOCHI UNIVERSITY OF TECHNOLOGY

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Quantum fluid

Quantum fluid : Hydrodynamic systems dominated by quantum effect and statistics

Quantum fluid

Equilibrium

- ^{4}He (Bose)
- ^{3}He (Fermi)
- Superconductors (Fermi)
- Atomic BEC (Bose)
- Fermionic BEC-BCS (Fermi)
- Neutron stars (Fermi)

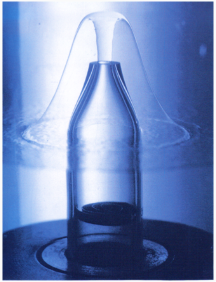
Out of equilibrium

- Exciton-polariton BEC (Fermi)
- Magnon BEC (Bose)

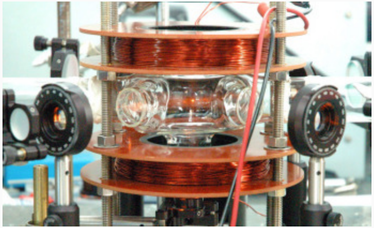
Superfluidity and quantized vortices

Superfluidity : Inviscid flows in macroscopic scales

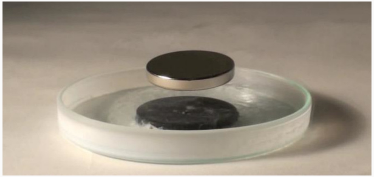
Superfluid He



Atomic BEC

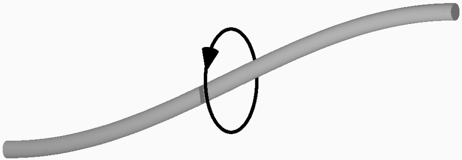


Superconductors



Quantized vortices

Rotational flows are carried by quantized vortices



$$\kappa = \oint d\boldsymbol{\ell} \cdot \boldsymbol{v} = h/m$$

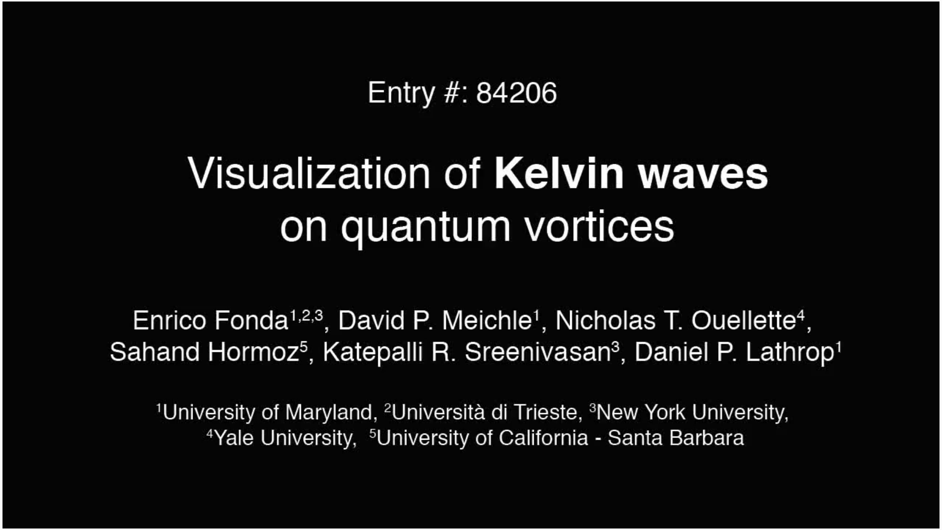
- Quantization of circulation (κ)
- Topological defects
- Very thin cores

  $\sim$ Å : ^{4}He

  $\sim$ 10 nm : ^{3}He

  $\sim$ 100 nm : Atomic BEC

Youtube video by groups in University of Maryland



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Macroscopic wave function and superfluidity

Superfluidity is triggered by long-range order in Bose-Einstein condensation

$\Rightarrow$ Complex order parameter ψ

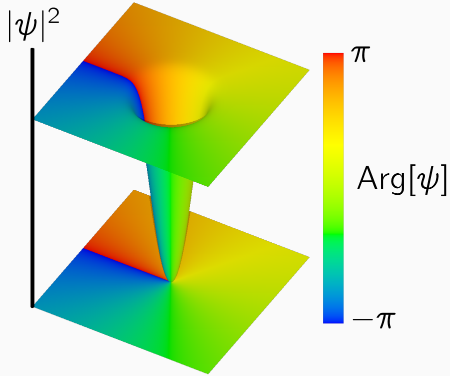
Superfluid density ρ and superfluid velocity $\boldsymbol{v}$

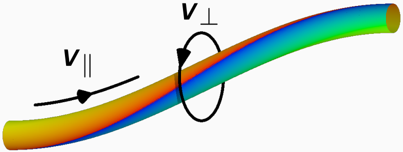
$$\psi = |\psi| \exp(i \mathrm{Arg}[\psi])$$

$$\rho = |\psi|^2 \qquad \boldsymbol{v} = \frac{\hbar}{m} \nabla \mathrm{Arg}[\psi]$$

Quantized vortices

Vortices as topological defects





$$|\boldsymbol{v}|_\perp \propto \frac{1}{|\boldsymbol{x}-\boldsymbol{x}_0|}$$

$$\oint d\boldsymbol{\ell}\cdot\boldsymbol{v}_\perp = \frac{h}{m} \equiv \kappa$$

Gross-Pitaevskii equation

Many-body Schrödinger equation

$$i\hbar\dot{\Psi} = \left[-\sum_{i=1}^{N} \frac{\hbar^2}{2m}\nabla_i^2 + \sum_{N_B=2}^{N} \frac{1}{N_B!} \sum_{i_1\neq\cdots\neq i_{N_B}} V^{(N_B)}(\boldsymbol{x}_{i_1}, \cdots \boldsymbol{x}_{i_{N_B}})\right] \Psi$$

Mean-field approximation

$$\Psi(\boldsymbol{x}_1, \cdots, \boldsymbol{x}_N) \approx \prod_{i=1}^{N} \psi(\boldsymbol{x}_i)$$

Local interaction approximation

$$V^{(N_B)}(\boldsymbol{x}_{i_1}, \cdots \boldsymbol{x}_{i_{N_B}}) = g^{(N_B)} \prod_{i_a\neq i_b} \delta(\boldsymbol{x}_{i_a} - \boldsymbol{x}_{i_b})$$

Gross-Pitaevskii equation : hydrodynamic equation for ψ

$$i\hbar\dot{\psi} = \left\{-\frac{\hbar^2}{2m}\nabla^2 + V(|\psi|^2)\right\}\psi$$

Gross-Pitaevskii equation

Gross-Pitaevskii equation : hydrodynamic equation for ψ

$$i\hbar\dot{\psi} = \left\{-\frac{\hbar^2}{2m}\nabla^2 + V(|\psi|^2)\right\}\psi$$

Hydrodynamic form (Quantum Euler equation)

$$E_{\mathrm{GP}} = \int d\boldsymbol{x}\,\left\{\frac{\rho\boldsymbol{v}^2}{2} + \frac{\hbar^2}{2m}(\nabla\sqrt{\rho})^2 + U(\rho)\right\} \qquad U = \int d\rho\, V : \text{Internal energy}$$

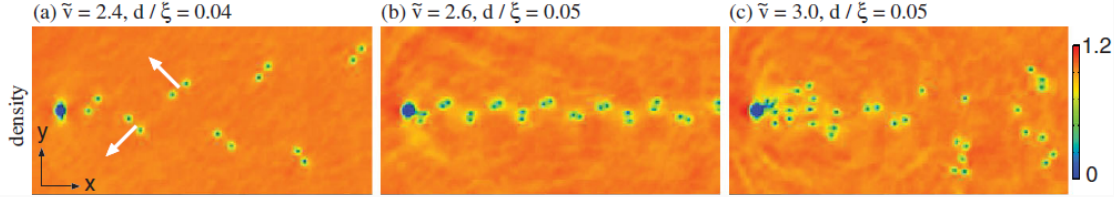
$$\dot{\rho} + \nabla\cdot(\rho\boldsymbol{v}) = 0 \qquad \dot{\boldsymbol{v}} + \frac{1}{2}\nabla\boldsymbol{v}^2 = -\frac{1}{m}\frac{\nabla p}{\rho} + \frac{\hbar^2}{2m}\nabla\left(\frac{\nabla^2\sqrt{\rho}}{\sqrt{\rho}}\right)$$

$$\frac{\nabla p}{\rho} = \frac{\partial\nabla U}{\partial\rho} : \text{Classical pressure} \qquad \nabla\left(\frac{\nabla^2\sqrt{\rho}}{\sqrt{\rho}}\right) : \text{Quantum pressure}$$

Vortex dynamics described by Gross-Pitaevskii equation

Vortex nucleation behind obstacle ⇒ dissipation

K. Sasaki et. al., PRL **104** 150404 (2010)



$$i\hbar\dot{\psi} = \left\{-\frac{\hbar^2}{2m}\nabla^2 + g|\psi|^2 + V_{\text{obstacle}}(\boldsymbol{x})\right\}\psi \qquad \psi(\boldsymbol{x})|_{t=0} = \sqrt{\rho}e^{imvx/\hbar}$$

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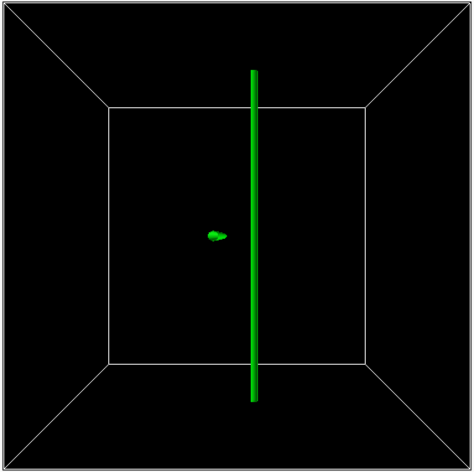
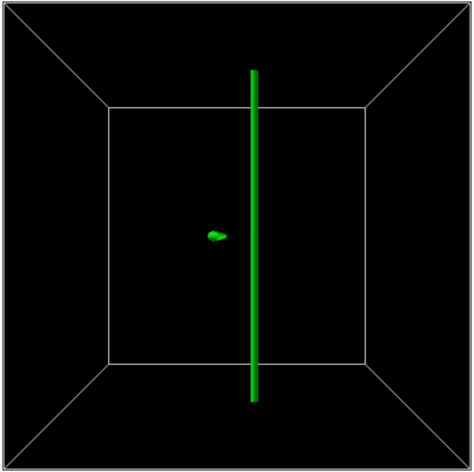
Vortex reconnection



- Two vortices approach to each other
- Reconnect at local antiparallel point
  : Topological change of structure
- Excitation of spiral Kelvin waves

Vortex reconnection



- Two vortices approach to each other
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Time-reversal symmetry of Gross-Pitaevskii equation

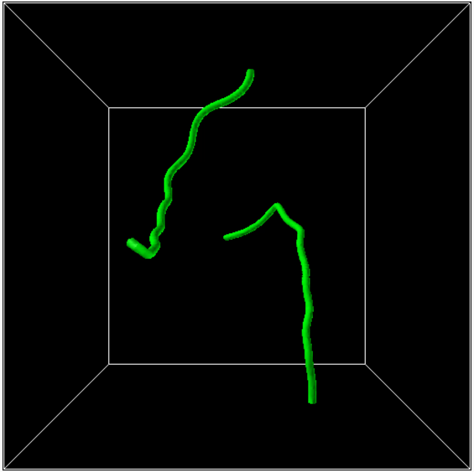
$$i\dot{\psi} = \left\{ -\frac{\hbar^2}{2m}\nabla^2 + V(|\psi|^2) \right\} \psi$$

$$\Downarrow t \to -t \quad \psi \to \psi^*$$

$$i\dot{\psi}^* = \left\{ -\frac{\hbar^2}{2m}\nabla^2 + V(|\psi|^2) \right\} \psi^*$$

ψ and ψ^* evolve along opposite time direction

Inverse reconnection of vortices



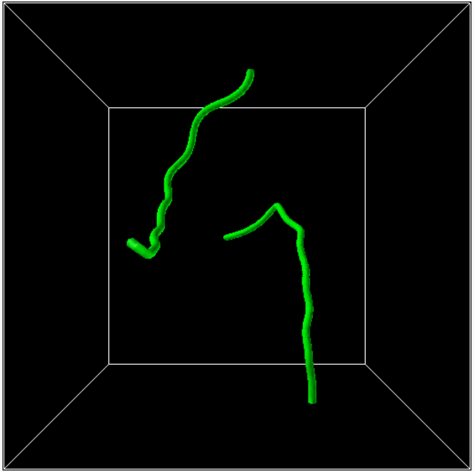
Time reversal Gross-Pitaesvkii equation

$$i\dot{\psi} = \left\{-\frac{\hbar^2}{2m} + V(|\psi|^2)\right\}\psi$$

$$\Downarrow \; t \to -t \quad \psi \to \psi^*$$

$$i\dot{\psi}^* = \left\{-\frac{\hbar^2}{2m} + V(|\psi|^2)\right\}\psi^*$$

Inverse reconnetion of vortices



Time reversal Gross-Pitaesvkii equation

$$i\dot{\psi} = \left\{-\frac{\hbar^2}{2m} + V(|\psi|^2)\right\}\psi$$

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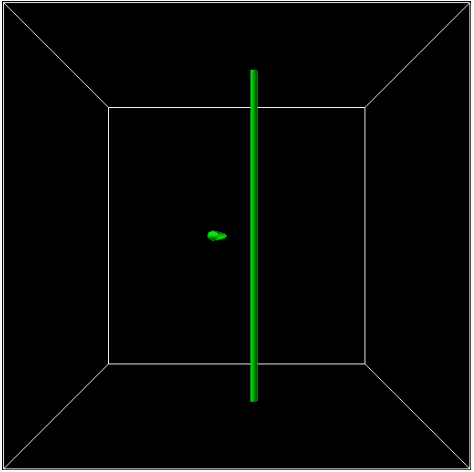
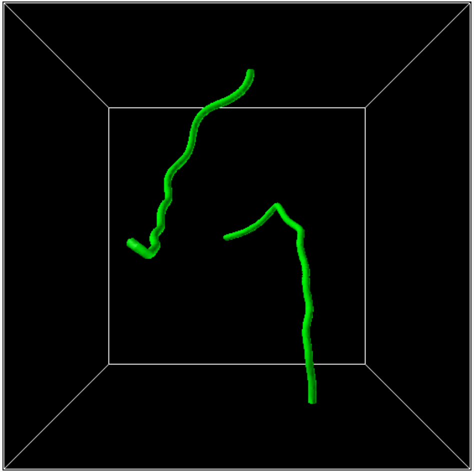
Time reversal symmetry allows both directions of reconnection

Forward reconnection



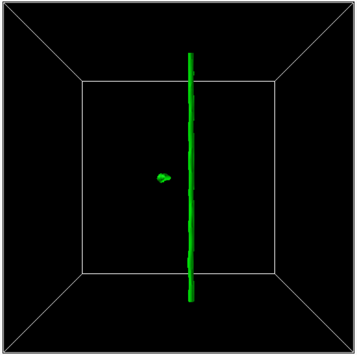
Backward reconnection



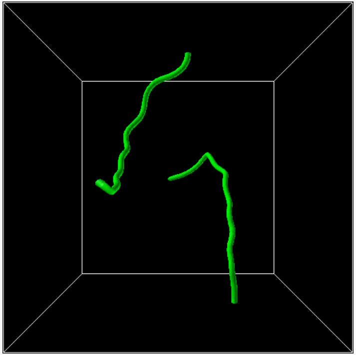
Difference between “forward” and “backward” reconnections

Adding white noise at $t = 0 : \psi(t = 0) \rightarrow \psi(t = 0) + \xi$

$\langle \xi^2 \rangle = 0.02$



$\langle \xi^2 \rangle = 0.01$



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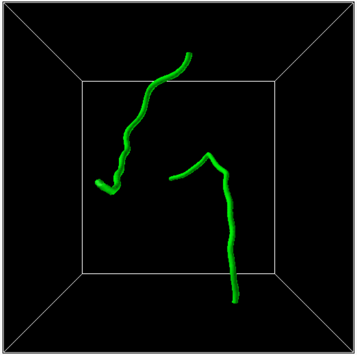

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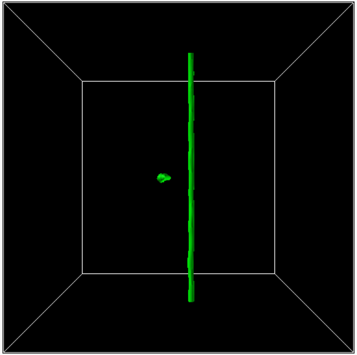

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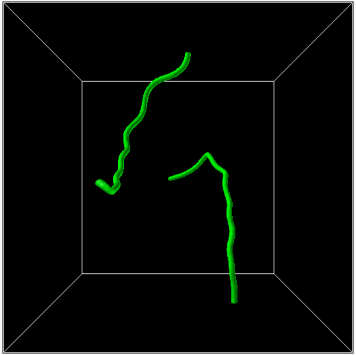

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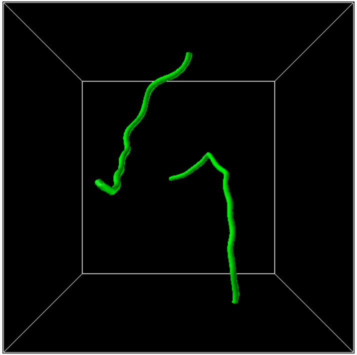
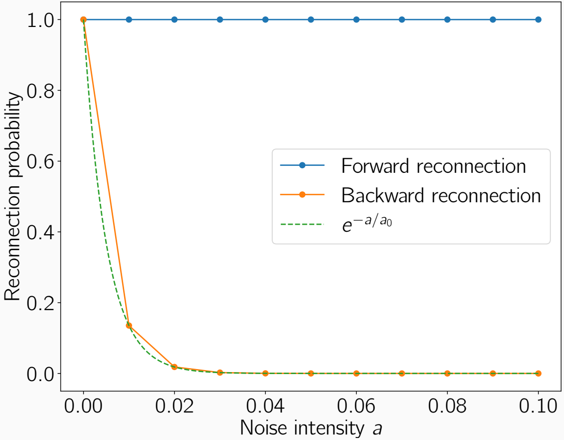


Probability of reconnection

$$\psi(\boldsymbol{x}, t=0) \to \psi(\boldsymbol{x}, t=0) + \xi \qquad \langle \xi \rangle = 0 \qquad \langle \xi^*(\boldsymbol{x}_1)\xi(\boldsymbol{x}_2)\rangle = a(\Delta x)^3 |\psi_\infty|^2 \delta(\boldsymbol{x}_1 - \boldsymbol{x}_2)$$



- Forward reconnection always occurs.
- Backward reconnection does not occur with small noise intensity a.
- Reconnection probability exponentially decays with a.

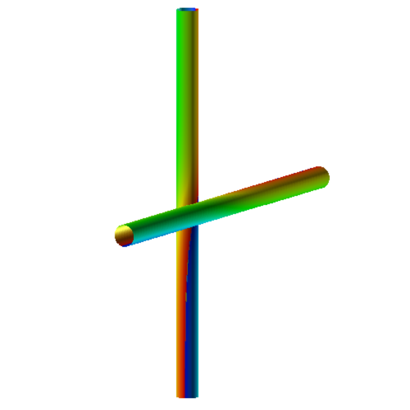
Intuitive properties of forward and backward reconnections

Reconnection type	Kelvin waves	Noisy initial condition
Forward	Emission	Robust
Backward	Absorption	Weak

Vortex reconnection has “’*time arrow*”

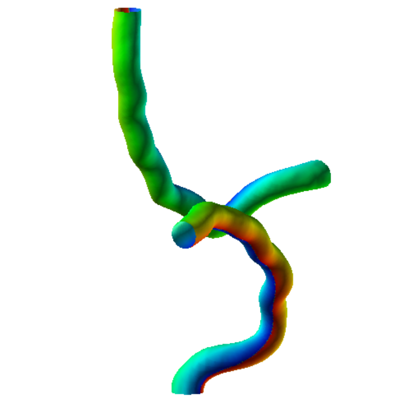
Phase structure around vortex core

Forward reconnection



Phase twist along vortex core

Backward reconnection



Less phase twist
(Kelvin waves instead)

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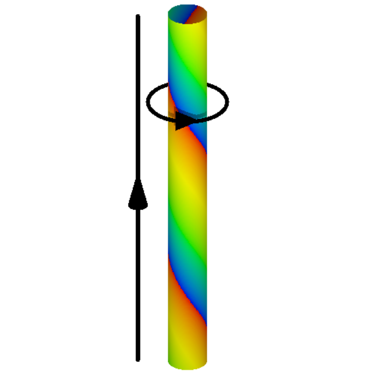
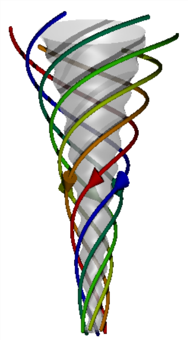
Phase twist along vortex core

Phase twist

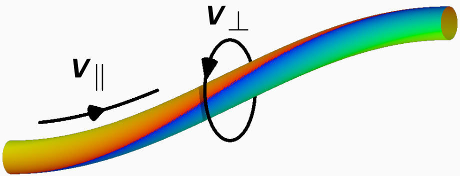


Correspondence in classical fluid



Corkscrew-like flow

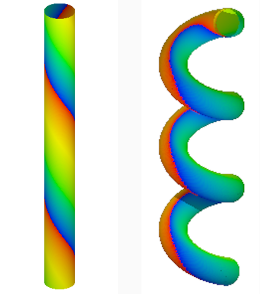
Phase twist is a part of centerline helicity



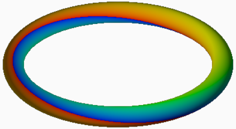
Centerline helicity

$$H = \int d\boldsymbol{x}\, \boldsymbol{v}_{\parallel} \cdot (\nabla \times \boldsymbol{v}_{\perp})$$

Centerline helicity = Phase twist + Torsion of vortex



All vortices have forms of closed loops



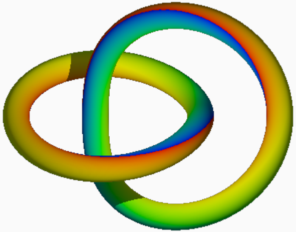
A simple ring cannot have the phase twist

Phase twist in bulk : knot and link

All vortices have forms of closed loops

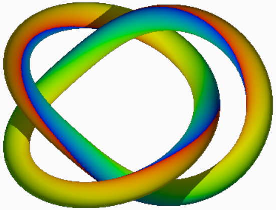
Link $(2-2$ torus knot)

$\Rightarrow$ Phase twist 4π



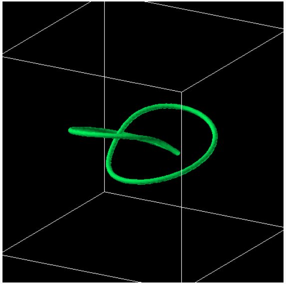
Knot $(2-3$ torus knot)

$\Rightarrow$ Phase twist 6π

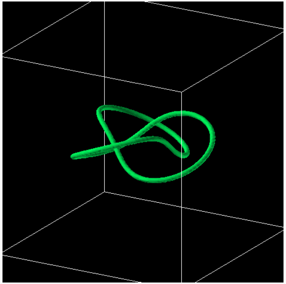


Reconnections of link and knot

Link (2 – 2 torus knot)



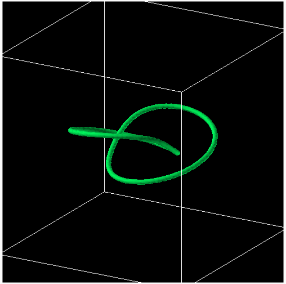
Knot (2 – 3 torus knot)



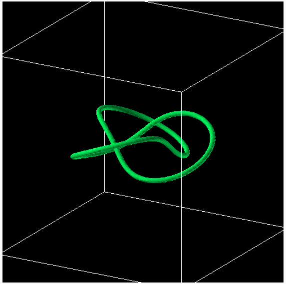
Links and knots having nonzero phase twist always reconnect

Reconnections of link and knot

Link (2 – 2 torus knot)



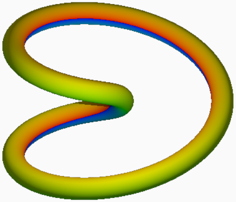
Knot (2 – 3 torus knot)



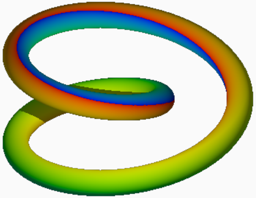
Links and knots having nonzero phase twist always reconnect

Writhe (2 – 1 torus knot)

Writhing number 0 ⇒ zero phase twist



Writhing number 1 ⇒ nonzero phase twist



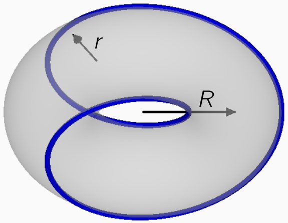
Solid angle for vortex crossing $< 2\pi \;\Rightarrow\;$ phase twist 0

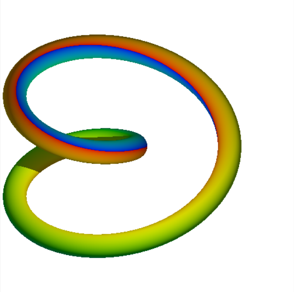
Solid angle for vortex crossing $> 2\pi \;\Rightarrow\;$ phase twist 2π

$2-1$ torus knot and phase twist

Ratio between
major radius R and minor radius r
of torus :

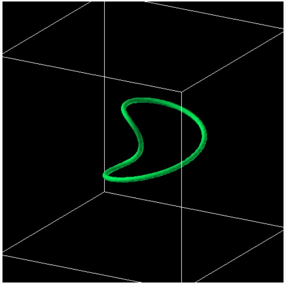
$R/r < 2 \Rightarrow$ phase twist 0
$R/r > 2 \Rightarrow$ phase twist 2π



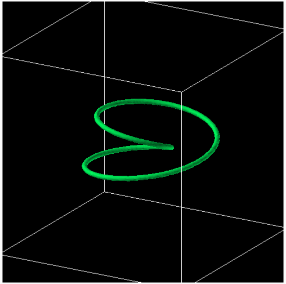


Reconnection of writhe

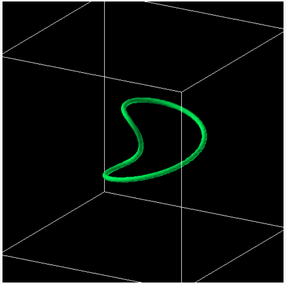
Phase twist 0



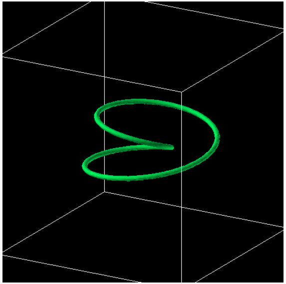
Phase twist 2π



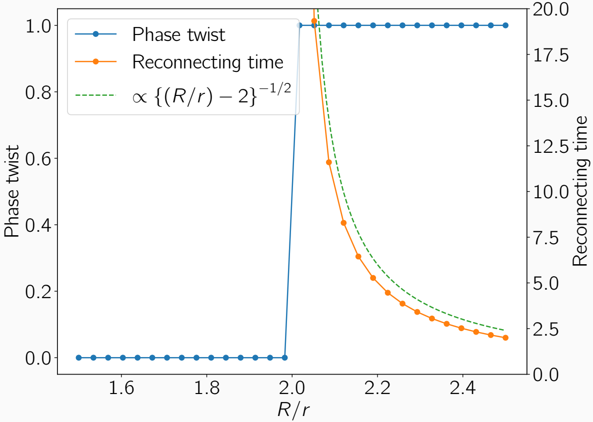
Phase twist 0



Phase twist 1



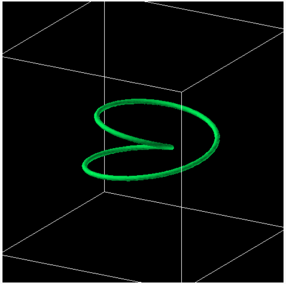
Reconnection time of writhe



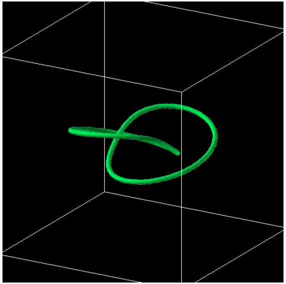
Initial condition having zero phase twist does not reconnect

Summary for closed loops

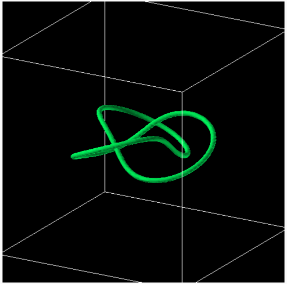
Writhe (2π)



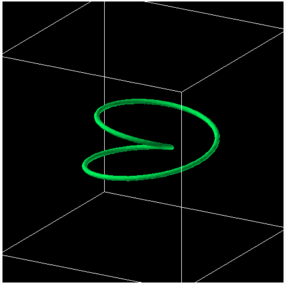
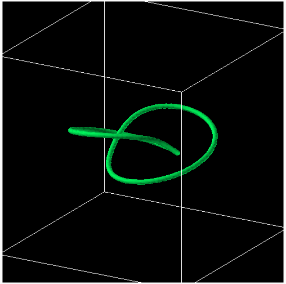
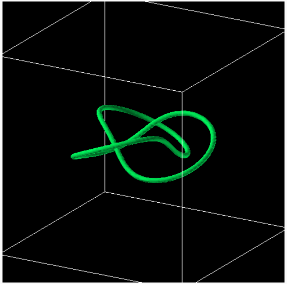
Link (4π)



Knot (6π)

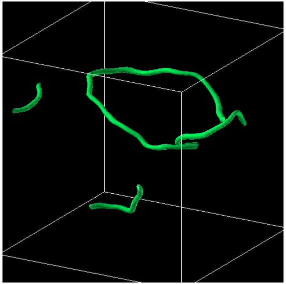


Summary for closed loops

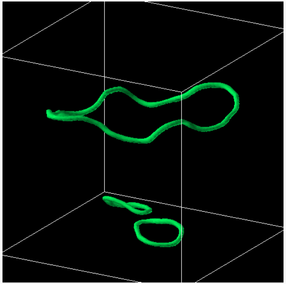
Writhe (2π)	Link (2π)	Knot (2π)
		

Time reversal dynamics (Adding noise)

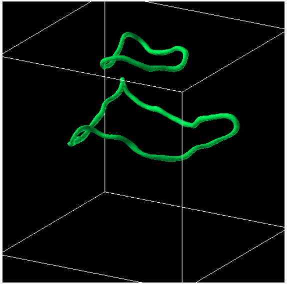
Writhe



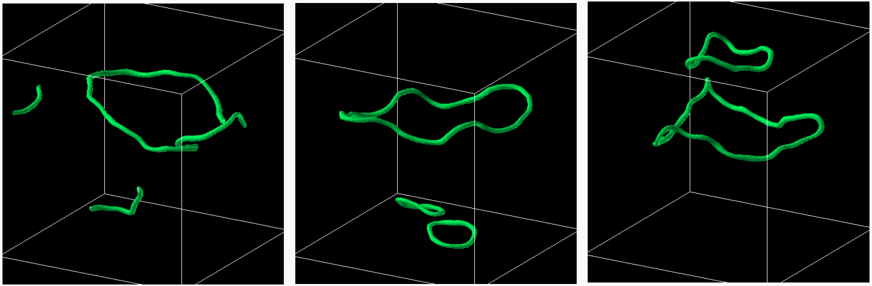
Link



Knot



Writhe Link Knot



Outline

Phase twist, torsion, centerline helicity

$$\text{Centerline helicity}: H = \int d\boldsymbol{x}\, \boldsymbol{v}_{\parallel} \cdot (\nabla \times \boldsymbol{v}_{\perp}) = \text{Phase twist} + \text{Vortex torsion}$$

Rewriting by the integral along vortex line (Euler's integral)

$$H = \frac{\kappa^2}{4\pi} \sum_{i,j:\text{closed loop}} \oint_{C_i} \oint_{C_j} \frac{(\boldsymbol{x}_i - \boldsymbol{x}_j) \cdot (d\boldsymbol{x}_i \times d\boldsymbol{x}_j)}{|\boldsymbol{x}_i - \boldsymbol{x}_j|^2} = H_{i=j} + H_{i \neq j}$$

$$H_{i \neq j} = \kappa^2 N_{\text{link}} \Rightarrow \text{Linking number}$$

$$H_{i=j} = \kappa^2 (N_{\text{knot}} + N_{\text{writhe}} + T_{\text{torsion}}) \Rightarrow \text{Knot+Writhe+Torsion}$$

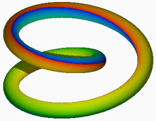
$$N_{\text{writhe,link,knot}} \in \mathbb{Z} \qquad T_{\text{torsion}} \in \mathbb{R}$$

Reconnection : interchange between $N_{\text{writhe,link,knot}}$ and T_{torsion}

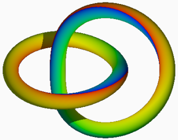
Centerline helicity of torus knot

$$H = \frac{\kappa^2}{4\pi} \sum_{i,j:\text{closed loop}} \oint_{C_i} \oint_{C_j} \frac{(\boldsymbol{x}_i - \boldsymbol{x}_j) \cdot (d\boldsymbol{x}_i \times d\boldsymbol{x}_j)}{|\boldsymbol{x}_i - \boldsymbol{x}_j|^2} = H_{i=j} + H_{i \neq j}$$

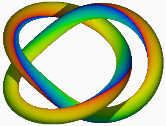
Writhe

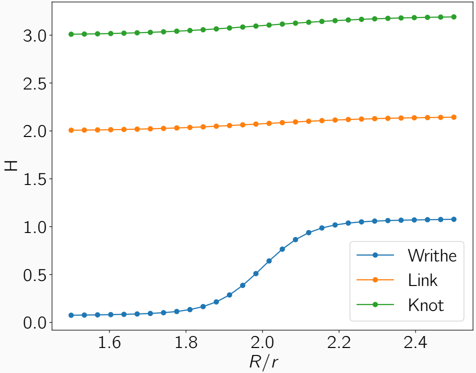


Link



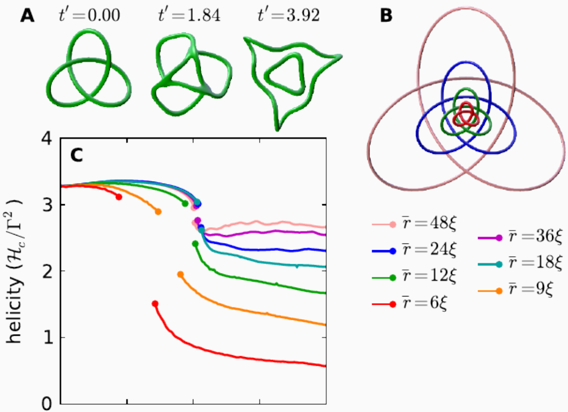
Knot





Note : Centerline helicity does not conserve in GP equation

M. W. Scheeler, et. al, PNAS **111**, 15350 (2014)



Quantum pressure breaks the conservation law.

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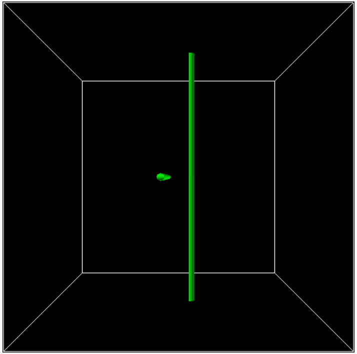
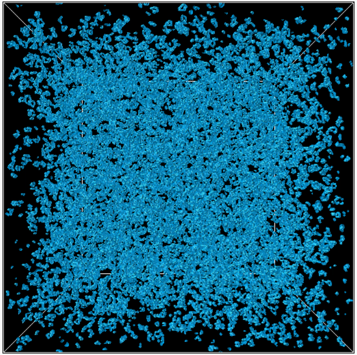
Phase twist and time arrow

Reconnection has time arrow in the time-reversal dynamics
$\Rightarrow$ Nature likes vortex torsion (Kelvin-waves) rather than phase twist.



Phase twist descreases in “normal” dynamics
$\Rightarrow$ Similar to entropy



Does equilibrium have phase twist?

Stochastic Gross-Pitaevskii equation : model for superfluid at finite temperatures

MK and L. F. Cugliandolo, PRE **94**, 062146 (2016)

Observable $F[\psi]$

$$\langle F \rangle = \frac{\int D\psi \, D\psi^* \, F e^{-\beta E_{\rm GP}}}{\int D\psi \, D\psi^* \, e^{-\beta E_{\rm GP}}} \qquad E_{\rm GP} = \int d\boldsymbol{x} \left\{ \frac{\hbar^2}{2m} |\nabla \psi|^2 + U(|\psi|^2) \right\}$$

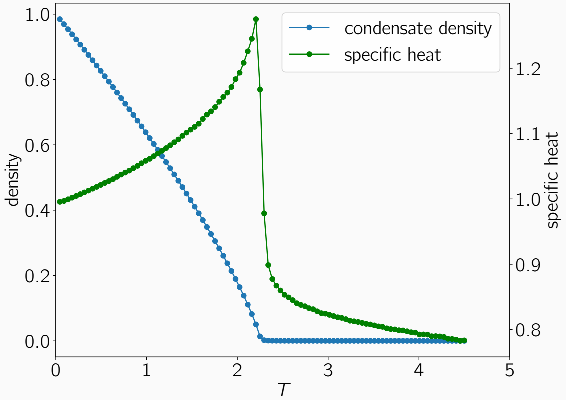
$$\Downarrow$$

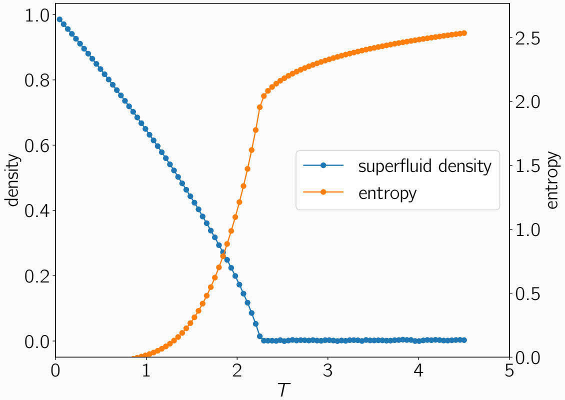
Stochastic Gross-Pitaevskii equation

$$(i\hbar - \gamma)\dot{\psi} = \left\{ -\frac{\hbar^2}{2m} \nabla^2 - \mu + V(|\psi|^2) \right\} \psi + \xi_1 + i\xi_2$$

$$\langle \xi_i(\boldsymbol{x}, t) \xi_j(\boldsymbol{x}', t') \rangle = \gamma T \delta(\boldsymbol{x} - \boldsymbol{x}') \delta(t - t') \delta_{ij} \qquad \langle F \rangle = \lim_{t \to \infty} \frac{1}{t} \int_0^t dt \, F$$

Observable for physical quantities

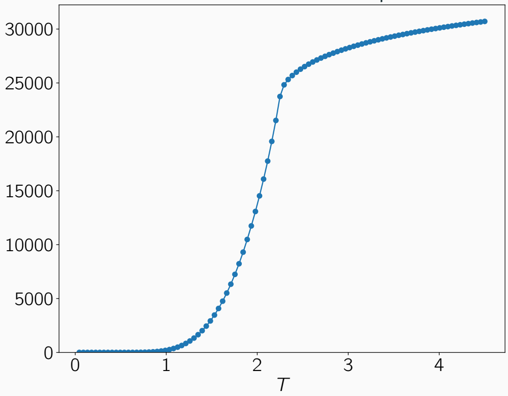




Stochastic GP equation can describe equilibrium state including phase transition for Bose-Einstein condensation

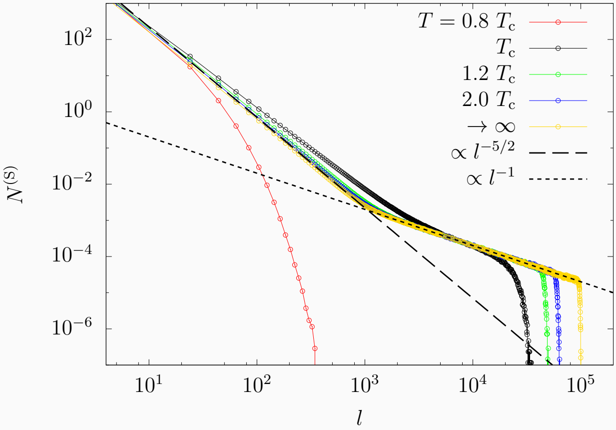
Observable for vortex quantities

Number of vortex loops



Number of loops increases with the temperature

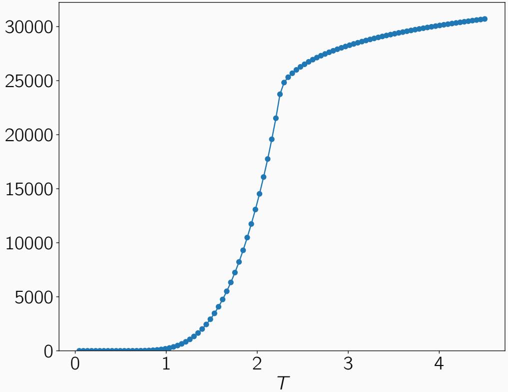
Size distribution of loop length



Long loop appears at hight temperatures

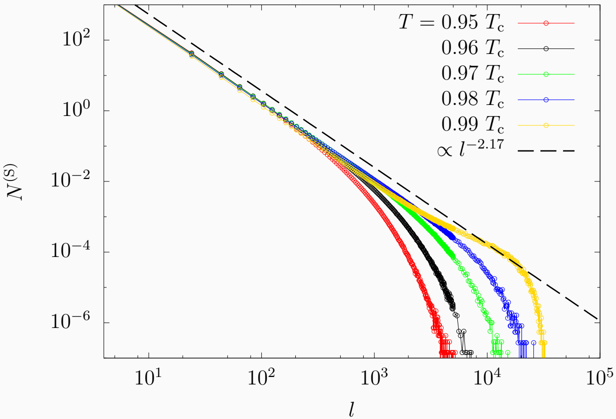
Observable for vortex quantities

Number of vortex loops



Number of loops increases with the temperature

Size distribution of loop length



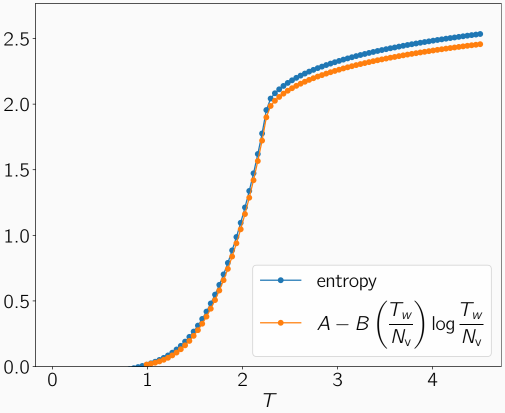
Long loop appears at hight temperatures

Phase twist T_w, number of vortex loops N_v and entropy

Entropy : $S = \frac{\langle E \rangle - F}{T}$ $\quad F = -T \langle \log e^{-E/T} \rangle$

T_w : Total twist for all vortex loops

N_v : Total number of vortex loops



$$-\left(\frac{T_w}{N_v}\right) \log \frac{T_w}{N_v}$$ behaves like thermodynamic entropy

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Summary

- Vortex reconnection has “time arrow” in time-reversal dynamics
- Direction of time arrow is determined by phase twist along vortex line
- Phase twist tends to decrease in vortex reconnections
- Phase twist behaves like entropy in equilibrium

  $\Rightarrow$ Phase twist can be expected to give “nonequilibrium entropy”

- What is the relation of phase twist to the fluctuation theorem?

$$\frac{P(\Delta S/\tau = A)}{P(\Delta S^{\dagger}/\tau = -A)} = e^{A\tau}$$