Mathematical studies on the finite temperature BEC model

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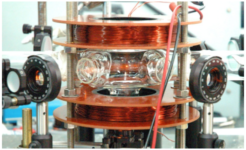
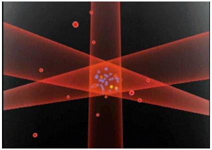
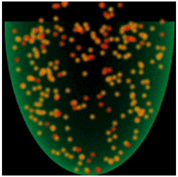
Bose-Einstein Condensation

A state of matter of a dilute gas of bosons cooled to temperatures very close to absolute zero; a large amount of bosons occupy the lowest quantum state, where macroscopic quantum phenomena can be observed.
Predicted in 1924-25, Realization in 1995.







^{87}Rb, ^{23}Na, ^{7}Li, ^{1}H, ^{85}Rb, ^{41}K, ^{4}He, ^{133}Cs, ^{174}Yb, ^{52}Cr, ^{40}Ca, ^{84}Sr : Example of the dilute gases of bosons

Modeling

- $x_1, \ldots, x_N$ particles in a trapping potential V, two-body interactions

$$\hat{H} = \sum_{j=1}^{N} \left(-\frac{\hbar^2}{2m} \Delta_{x_j} + V(x_j) \right) + \sum_{1 \le j,k \le N} U(x_j - x_k)$$

- Ground state: minimum of energy corresponding to $\hat{H}$ for the wave function $\tilde{\psi}(x_1, .., x_N)$
- For very small temperature T, thermal wavelength of particles

$$\lambda_T = \frac{\hbar}{(2\pi m k_B T)^{1/2}}$$

  is larger than the particle distance
- Replace interaction potential U by

$$U_{eff}(x) = \frac{4\pi\hbar^2 a}{m} \delta_0(x)$$

  a: atomic diffusion length (positive or negative)

Modeling

- Bose gaz: Hatree approximation

$$\tilde{\psi}(x_1, x_2, ..., x_N) = \prod_{j=1}^{N} \psi(t, x_j)$$

- large number of particles - rescaling
- Gross-Pitaevskii (1961, superfluids)

$$i\hbar\partial_t\psi(t,x) = \left(-\frac{\hbar^2}{2m}\Delta + V(x) + \frac{4\pi\hbar^2 a}{m}|\psi|^2\right)\psi := L_{GP}\psi$$

V: confinement potential

Non zero temperature

- Aim: modeling of condensate close to critical temperature $T_{cr} > 0$
- need modeling of interactions with non condensated atoms, here assumed "thermalized"
- ψ: wave function for the condensated atoms

$$L_{GP} = -\frac{\hbar^2}{2m}\Delta + V(x) + \frac{4\pi\hbar^2 a}{m}|\psi|^2$$

where m is the mass, a is the positive s-wave scattering length.

- Stochastic projected GPE: Duine, Stoof 2001 Gardiner-Davis, 2003

$$d\psi = \mathcal{P}_c\Big\{ -\frac{i}{\hbar}L_{GP}\psi dt + \frac{\gamma}{k_B T}(\nu - L_{GP})\psi dt + dW(x,t)\Big\}$$

where $\mathcal{P}_c$: projection to the lowest energy modes, ν: chemical potential

$$\langle dW(s,y), dW(t,x)\rangle = 2\gamma\delta_{t-s}\delta_{x-y}$$

Additional terms: interaction thermal cloud-condensate

Equilibrium state

Energy: ψ_c projected wave function (d_c-dimensional)

$$H(\psi_c) = \frac{\hbar^2}{2m}|\nabla\psi_c|_{L^2}^2 + \frac{1}{2}|\sqrt{V(x)}\psi_c|_{L^2}^2 - \frac{\nu}{2}|\psi_c|_{L^2}^2 + \frac{1}{4}|\psi_c|_{L^4}^4,$$

with

$$V(x) = \frac{m}{2}\omega^2|x|^2$$

- Ground state ($T = 0$) : (when ν large) global minimum, thus stable
- Gibbs mesure ($T > 0$):

$$\rho_T(d\psi_c) = \alpha_c \exp\left(-\frac{H(\psi_c)}{k_B T}\right) d\psi_c$$

  - Spontaneous nucleation of vortices in BEC (Weiler et al. Nature 2008).

Rigorously in mathematics,

- Treat the infinite dimension model
- Make sense the Gibbs measure and the convergence of the system

Infinite dimensional model

- $(\Omega, \mathcal{F}, \mathbb{P})$: probability space endowed with filtration $(\mathcal{F}_t)_{t\geq 0}$
- Write the equation in dimensionless form :

$$d\Psi = (i+\gamma)(\Delta\Psi - V(x)\Psi + \nu\Psi - \lambda|\Psi|^2\Psi)dt + \sqrt{2\gamma}dW$$

where $\gamma > 0$. Assume $V(x) = |x|^2$, $\nu \geq 0$ and $\lambda = 1$ (defocusing).

- $Ah_k = (-\Delta + |x|^2)h_k = \lambda_k^2 h_k$ with $\lambda_k^2 = 2|k| + d$, $k \in \mathbb{N}^d$
  $h_k(x)$: Hermite functions.
- $W(t)$: cylindrical Wiener pocess on $L^2(\mathbb{R}^d, \mathbb{C})$, i.e.

$$W(t,x) = \sum_{k\in\mathbb{N}^d} \beta_k(t)h_k(x), \quad t > 0, \ x \in \mathbb{R}^d$$

where $\{\beta_k(t)\}_{k\in\mathbb{N}^d}$: a seq. of $\mathbb{C}$-valued independent BM.

$$\mathbb{E}(W(t,x)W(s,y)) = (s\wedge t)\sum_k h_k(x)h_k(y),$$

$$\int_{\mathbb{R}^d}\sum_k h_k(x)h_k(y)\phi(y)dy = \sum_k h_k(x)(h_k,\phi)_{L^2} = \phi(x) = \int_{\mathbb{R}^d}\delta(x-y)\phi(y)dy.$$

Gibbs measure

- Constructive quantum field theory;
  Simon, Lieb... 60's
- Lebowitz-Rose-Speer 1988, Bourgain 1994: Gibbs measures and global existence for dispersive equations (Hamiltonian systems) ; lots of results since then Burq, Gerard, Thomann, Tzvetkov, Colliander, Oh, Bourgain, Bulut,...
- Stochastic case (BEC model): Carlen-Fröhlich-Lebowitz 2017 (regular noise), De Bouard-Debussche-F. 2018, (Hoshino 2018)
- Stochastic case (Φ^4 model): Da Prato-Debussche 2003, Tsatsoulis-Weber 2018, Albeverio-Kusuoka 2020, Gubinelli-Hofmanova 2021, Oh-Okamoto-Tolomeo...

The case $d = 1$.

- Hamiltonian
$$H(\psi) = \frac{1}{2}\langle \psi, A\psi \rangle + \frac{1}{4}\int_{\mathbb{R}} |\psi|^4 dx$$
where $A = -\partial_x^2 + x^2$ with eigenvalues $\lambda_k^2 = 2k+1$, $k \in \mathbb{N}$.
- We may formally write
$$\begin{aligned} \rho(d\psi) &= \Gamma^{-1} e^{-H(\psi)} d\psi \\ &= \Gamma^{-1} e^{-\frac{1}{4}\int_{\mathbb{R}} |\psi|^4 dx} e^{-\frac{1}{2}\langle A\psi, \psi \rangle} d\psi. \end{aligned}$$
- The last term may be written using the decomposition $\psi = \sum_k (a_k + ib_k) h_k$ with $(a_k, b_k) \in \mathbb{R}^2$,
$$\prod_k \frac{\lambda_k^2}{2\pi} e^{-\frac{\lambda_k^2}{2}(a_k^2 + b_k^2)} da_k db_k, \text{ and this is a Gaussian measure.}$$
We call the limit ρ as $N \to \infty$ the Gibbs measure if exists:
$$\rho := \lim_{N \to \infty} \Gamma_N^{-1} e^{-\frac{1}{4}\int_{\mathbb{R}} |\psi_N|^4 dx} e^{-\frac{1}{2}\langle A\psi_N, \psi_N \rangle} d\psi_N$$
ψ_N is a finite dimensional cut of ψ.

- The Gaussian measure ($:= \mu$) is equivalent to the law of

$$Z_\gamma(t) = \sqrt{2\gamma} \int_{-\infty}^{t} e^{-(t-s)(i+\gamma)(-\partial_x^2 + x^2)} dW(s),$$

the stationary solution of

$$dZ = (i+\gamma)(\partial_x^2 - x^2) Z dt + \sqrt{2\gamma} dW$$

- Write $Z_\gamma(t)$ using the basis $\{h_k\}_k$,

$$Z_\gamma(t) = \sum_{k \in \mathbb{N}} \left(\sqrt{2\gamma} \int_{-\infty}^{t} e^{-(t-s)(i+\gamma)\lambda_k^2} d\beta_k(s) \right) h_k$$

- (in 1d) It is known the decay of h_k in L^p (Koch-Tataru, Duke Math.J. 2005): for $p \geq 4$, $|h_k|_{L^p(\mathbb{R})} \leq C_p \lambda_k^{-1/6}$, and by interpolation, if $2 \leq p \leq 4$, $|h_k|_{L^p(\mathbb{R})} \leq C_p \lambda_k^{-\frac{1}{3}(1-\frac{2}{p})}$.
- Recall that $\lambda_k^2 = 2k+1$ and thus the series converges for $p > 2$, i.e.,

$$Z_\gamma \in L^{2m}(\Omega; L^p) \quad \text{for any} \quad m \geq p/2 > 1$$

i.e. $Z_\gamma \in L^p$ a.s. i.e. $\mu(L^p) = 1$ for $p > 2$, thus $\rho(L^p) = 1$.

Let $p \geq 3$, $\Psi(0) \in L^p(\mathbb{R})$, $\gamma > 0$ and $\nu = 0$ (Theorems hold also for $\nu > 0$).

Theorem (de Bouard, Debussche, F.(2018))

There exists a set $\mathcal{O} \subset L^p(\mathbb{R})$ such that $\rho(\mathcal{O}) = 1$, and such that for $\Psi(0) \in \mathcal{O}$ there exists a unique solution $\Psi(\cdot) \in C([0,\infty), L^p(\mathbb{R}))$ a.s.

$P_t\phi(y) := \mathbb{E}(\phi(\Psi(t,y)))$, $y \in \mathcal{O}$, $t \geq 0$.

Theorem (de Bouard, Debussche, F.(2018))

Let $\phi \in L^2((L^p, d\rho), \mathbb{R})$, and $\bar{\phi} = \int_{L^p} \phi(y) d\rho(y)$. Then $P_t\phi(\cdot)$ converges exponentially to $\bar{\phi}$ in $L^2((L^p, d\rho), \mathbb{R})$, as $t \to \infty$; more precisely,

$$\int_{L^p} |P_t\phi(y) - \bar{\phi}|^2 d\rho(y) \leq e^{-\gamma t} \int_{L^p} |\phi(y) - \bar{\phi}|^2 d\rho(y).$$

Using Strong Feller property + Irreducibility of P_t on L^p,

Theorem (de Bouard, Debussche, F.(2018))

For any $\Psi(0) \in L^p(\mathbb{R})$, there exists a unique solution $\Psi(\cdot) \in C([0,\infty), L^p(\mathbb{R}))$ a.s.

The case $d = 2$.

$$d\Psi = (i+\gamma)(\Delta\Psi - |x|^2\Psi + \nu\Psi - |\Psi|^2\Psi)dt + \sqrt{2\gamma}dW, \quad \gamma > 0.$$

- In 2d, the linear stationary sol. $Z_\gamma \in \mathcal{W}^{-s,q}(\mathbb{R}^2)$ if $s > 0$, $q \geq 2$, $sq > 2$ (thus, supp μ also).

$$\mathcal{W}^{\sigma,p}(\mathbb{R}^2) := \{f \in \mathcal{S}' : (-\Delta + |x|^2)^{\frac{\sigma}{2}} f \in L^p(\mathbb{R}^2)\}, \ \sigma \in \mathbb{R}, \ p > 1.$$

- (*Da Prato-Debussche trick*) decompose $\Psi = U + Z_\gamma =$(good regularity)+(bad regularity), U satisfies a random PDE:

$$\begin{aligned} \partial_t U &= (i+\gamma)(\Delta U - |x|^2 U - |U + Z_\gamma|^2(U + Z_\gamma)) \\ &= (i+\gamma)(\Delta U - |x|^2 U - |Z_\gamma|^2 Z_\gamma - 2U|Z_\gamma|^2 - U^2 \bar{Z}_\gamma \cdots) \end{aligned}$$

Wick products

- We define, for any $k, l \in \mathbb{N}$,

$$: Z_\gamma^k \bar{Z}_\gamma^l : \quad := \lim_{N \to \infty} H_{k,l}(S_N Z_\gamma; C_{V,N}^2)$$

where $S_N Z_\gamma = \sum_{k \in \mathbb{N}^2} \chi\left(\frac{\lambda_k^2}{\lambda_N^2}\right)(Z_\gamma, h_k)_{L^2} h_k$, χ:smooth cut-off

- $H_{k,l}(z;\sigma)$: complex Hermite polynomials: $H_{0,0}(z;\sigma) = 1$, $H_{1,0}(z;\sigma) = z$, $H_{2,0}(z;\sigma) = z^2$, $H_{1,1}(z;\sigma) = z\bar{z} - \sigma$, $H_{3,0}(z;\sigma) = z^3$, $H_{2,1}(z;\sigma) = z^2\bar{z} - 2\sigma z....$

$$C_{V,N}^2(x) = \mathbb{E}(|S_N Z_\gamma|^2) = \sum_{k \in \mathbb{N}^2} \chi\left(\frac{\lambda_k^2}{\lambda_N^2}\right) \frac{h_k^2(x)}{\lambda_k^2} = \sum_{k \in \mathbb{N}^2} \chi\left(\frac{\lambda_k^2}{\lambda_N^2}\right) \frac{h_k^2(x)}{2|k|+2}.$$

- The decay property in k of $h_k(x)$ is worse, compared to the previous torus cases. In particular the diverging 'constant' $C_{V,N}^2(x)$ is no more constant, and not $O(logN)$.
- For any $\{H_{k,l}(S_N Z_\gamma; C_{V,N}^2)\}_{N \in \mathbb{N}}$ is Cauchy in $L^q(\Omega, \mathcal{W}^{-s,q}(\mathbb{R}^2))$ with $s > 0$, $sq > 2$, $q > 2$.

Outline

Theorem (local existence, de Bouard-Debussche-F. (2022))

Fix any $T > 0$. *Let* $\gamma > 0$, $q > 18$, $0 < s < \frac{1}{9}$ *with* $qs > 2$. *Let* $U(0) \in \mathcal{W}^{-s,q}$. *Then there exist* $T_0^* = T_0^*(|U(0)|_{\mathcal{W}^{-s,q}}, \| : Z_\gamma^3 : \|_{\mathcal{W}^{-s,q}}(T)) > 0$ *and a unique solution* U *such that*

$$U \in C([0, T_0^*) : \mathcal{W}^{-s,q}) \cap L^r(0, T_0^* : \mathcal{W}^{\beta,p}) \quad a.s.,$$

for any β, p *and* r *satisfying* $q > p > 3r$, $r > 6$, $s < \beta < \frac{2}{p}$, $\beta - s > \frac{2}{p} - \beta$, *and* $s + 2\left(\frac{2}{p} - \beta\right) < 2\left(1 - \frac{1}{q}\right)$, *where*

$$\| : Z_\gamma^n : \|_{\mathcal{W}^{-s,q}}(T) = \max_{0 \le k,l \le n, k+l=n} \sup_{0 \le t \le T} | : Z_\gamma^k \overline{Z_\gamma^l} : |_{\mathcal{W}^{-s,q}}.$$

We have moreover almost surely $T_0^* = T$ *or* $\lim_{t \uparrow T_0^*} |U(t)|_{\mathcal{W}^{-s,q}} = +\infty$.

Estimates on the semi-group $e^{-t(i+\gamma)A}$: thanks to Mehler transform and estimates on the real GL semi-group: Let $\gamma > 0$. For small $t > 0$,

- $$|e^{-t(i+\gamma)A}f|_{L^r} \leq C_0 t^{-\frac{1}{l}}|f|_{L^s}, \quad f \in L^s(\mathbb{R}^2)$$

  with
  $$0 \leq \frac{1}{r} \leq \frac{1}{r} + \frac{1}{l} = \frac{1}{s} \leq 1.$$

- $$|e^{-t(i+\gamma)A}f|_{\mathcal{W}^{s,p}} \leq C_1 t^{-\frac{s}{2}}|f|_{L^p}, \quad f \in L^p(\mathbb{R}^2)$$

- Let $q > p > 1$ and $\sigma > \frac{1}{p} - \frac{1}{q}$.
  $$|e^{-t(i+\gamma)A}f|_{L^p} \leq C_2 t^{-\sigma}|f|_{L^q}, \quad f \in L^q(\mathbb{R}^2)$$

- (for the nonlinear terms) Let $1 < p, q < +\infty$, $0 < s < \beta < 2/p$, and $m \in \mathbb{N} \setminus \{0\}$. Suppose $\beta - s - (m-1)(\frac{2}{p} - \beta) > 0$, and $s + m(\frac{2}{p} - \beta) < 2(1 - \frac{1}{q})$.
  $$|hf^m|_{\mathcal{W}^{-(s+m(\frac{2}{p}-\beta)),q}} \leq C|h|_{\mathcal{W}^{-s,q}}|f|^m_{\mathcal{W}^{\beta,p}}.$$

- Let $2 < \tilde{q} < 2 + 2(\gamma^2 + \gamma\sqrt{1+\gamma^2})$. If $U \in C([0, T_0^*), L^{\tilde{q}})$ we have the a priori bound in $L^{\tilde{q}}$: for any $T > 0$ there exists a constant C such that

$$\sup_{0 \le t \le T_0^* \wedge T} |U(t)|_{L^{\tilde{q}}}^{\tilde{q}} \le e^{-\frac{\gamma \tilde{q} t}{8}} |U(0)|_{L^{\tilde{q}}}^{\tilde{q}} + C,$$

  where T_0^*: the maximal existence time in the above local theory, and C depends on $\gamma, \tilde{q}, \| : Z_\gamma^3 : \|_{\mathcal{W}^{-s,q}}(T)$.
- Choice $\tilde{q} = q \rightsquigarrow$ the restriction on γ (γ should be large)
- **Bootstrap argument** (Matsuda 2019): Using heat smoothing, we can upgrade the integrability and regularity of solution to have the enegy estimate for all $\gamma > 0$.

Theorem (global existence, de Bouard-Debussche-F. (2022))

Let $\gamma > 0$, $q > 18$, $0 < s < \frac{1}{9}$ with $qs > 2$. Let $U(0) \in \mathcal{W}^{-s,q}(\mathbb{R}^2)$. Then there exists a unique global solution U in $C([0, T], \mathcal{W}^{-s,q})$ a.s. for any $T > 0$.

Global existence for *any* $\gamma > 0$

Strategy (i)-(iv) :

(i) Starting from $U(0) \in \mathcal{W}^{-s,q}$, with $q > 18$, $0 < s < \frac{1}{9}$ with $qs > 2$, we first prove that for any $q_0 > 2$, and for any small $t_0 > 0$, we have $U(t_0) \in L^{q_0}$. The application of above $L^{\tilde{q}}$ bound with $\tilde{q} = q_0$ then show that U is uniformly bounded in L^{q_0}.

Use of heat smoothing + bilinear estimates

Lem. Let q_0 be such that $2 < q_0 < 2 + 2(\gamma^2 + \gamma\sqrt{1+\gamma^2})$. Then for any $0 < t_0 < t_1 < T_0^*$, $U \in C([t_0, t_1]; L^{q_0}(\mathbb{R}^2))$ and there is a constant $a > 0$ such that

$$\sup_{t \in [t_0, t_1]} |U(t)|_{L^{q_0}} \leq C\left(t_0^{-a}, |U(0)|_{\mathcal{W}^{-s,q}}, \| : Z_\gamma^3 : \|_{\mathcal{W}^{-s,q}}(T)\right).$$

(ii) We then show an estimate on $|U(t)|_{\mathcal{W}^{\gamma,p}}$ with $\gamma > 0$ small, and $p > 2$, close to 2.

- Heat smoothing + bilinear estimates + uniform L^{q_0} bound

Lem. There exists $p > 2$, $\gamma \in (0,1)$ such that for any $0 < t_0 < t_1 \leq T_0^* \wedge T$

$$\int_{t_0}^{t_1} |U(t)|^3_{\mathcal{W}^{\gamma,3p}} dt \leq C\Big(T, t_0^{-a}, |U(0)|_{\mathcal{W}^{-s,q}}, \| : Z_\gamma^3 : \|_{\mathcal{W}^{-s,q}}(T)\Big),$$

$$\sup_{t_0 \leq t \leq t_1} |U(t)|_{\mathcal{W}^{\gamma,p}} \leq C\Big(T, t_0^{-a}, |U(0)|_{\mathcal{W}^{-s,q}}, \| : Z_\gamma^3 : \|_{\mathcal{W}^{-s,q}}(T)\Big).$$

for some $a > 0$.

(iii) (bootstrap argument) Once we have such an estimate, by bootstrap arguments, we can upgrade the regularity from $\gamma > 0$ close to 0 to $\gamma < 1$ close to 1.

Lem. Assume that there exist $p > 2$, $\gamma \in (0, \frac{2}{p})$ such that for any $0 < t_0 < t_1 \leq T \wedge T_0^*$,

$$\int_{\frac{t_0}{2}}^{t_1} |U(t)|^3_{\mathcal{W}^{\gamma,3p}} dt + \sup_{\frac{t_0}{2} \leq t \leq t_1} |U(t)|_{\mathcal{W}^{\gamma,p}} \leq C.$$

Assume that $\varepsilon \in \left(0, \frac{5}{3} - \frac{10}{3p}\right)$ satisfies $\frac{\gamma+\varepsilon}{2} < \frac{5}{6} - \frac{2}{3p}$. Then,

$$\int_{t_0}^{t_1} |U(t)|^3_{\mathcal{W}^{\gamma+\varepsilon,3p}} dt + \sup_{t_0 \leq t \leq t_1} |U(t)|_{\mathcal{W}^{\gamma+\varepsilon,p}} \leq C'.$$

(iv) Then by the Sobolev embedding, we will obtain a bound on $|U(t)|_{\mathcal{W}^{-s,q}}$.

Lem. Now, repeating several times the bootstrap argument, one may thus show a uniform bound for U in $\mathcal{W}^{\gamma,p}$ for some p with $2 < p < q$ and any $\gamma < \frac{2}{p}$. Choosing then $\gamma = -s + 2\left(\frac{1}{p} - \frac{1}{q}\right)$, so that $\mathcal{W}^{\gamma,p} \subset \mathcal{W}^{-s,q}$, we obtain

$$\sup_{t_1 \le T \wedge T_0^*} \sup_{t \in [t_0,t_1]} |U(t)|_{\mathcal{W}^{-s,q}} \le C\left(T, t_0^{-a}, |U(0)|_{\mathcal{W}^{-s,q}}, \| : Z_\gamma^3 : \|_{\mathcal{W}^{-s,q}}(T)\right),$$

and global existence in $\mathcal{W}^{-s,q}$ follows.

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Existence of Gibbs measure

$$d\rho = \Gamma^{-1} e^{-\int_{\mathbb{R}^2} \frac{1}{4}|\psi|^4 dx} e^{-\frac{1}{2}\langle A\psi,\psi\rangle} d\psi$$

- We cannot make sense of $e^{-\int_{\mathbb{R}^2} \frac{1}{4}|\psi|^4 dx}$ since $L^4(\mathbb{R}^2)$ is not in $\mathrm{supp}\mu$. Need a renormalization both in the equation and in the Gibbs measure.

$$d\rho = \Gamma^{-1} e^{-\int_{\mathbb{R}^2} \frac{1}{4}:|\psi|^4:dx} e^{-\frac{1}{2}\langle A\psi,\psi\rangle} d\psi$$

- (D.Robert) We cannot define directly the limit $N \to \infty$ of $\exp\left\{-\frac{1}{4}\int_{\mathbb{R}^2} :|S_N\psi|^4:(x)dx\right\} d\mu_N(\psi)$, this problem is related to the property of the kernel of $(-\Delta + |x|^2)^{-1}$:

$$K(x,y) := \sum_{k\in\mathbb{N}^2} \frac{h_k(x)h_k(y)}{\lambda_k^2}, \text{ and } K \notin L^4_{x,y}.$$

- $\rightsquigarrow$ define the Gibbs measure as a limit of

$$\tilde{\rho}_N(d\psi) = \Gamma_N \exp\left\{-\int_{\mathbb{R}^2}\left(\frac{1}{4}|S_N\psi(x)|^4 - 2C_{V,N}^2(x)|S_N\psi(x)|^2 + 2C_{V,N}^4(x)\right)dx\right\}\mu_N(d$$

$$\psi \in E_N^{\mathbb{C}} = \mathrm{span}\{h_0, h_1, ..., h_N\}$$

- Let $\gamma > 0$. Considering a stationary solution $(U_N, Z_N) \in C(\mathbb{R}_+; E_N^{\mathbb{C}} \times E_N^{\mathbb{C}})$ of the coupled evolution on $E_N^{\mathbb{C}}$ given by

$$\begin{cases} \partial_t U & = \gamma \left[HU - S_N(: |S_N(U+Z)|^2 S_N(U+Z) :) \right] \\ dZ & = \gamma HZdt + \sqrt{2\gamma}\Pi_N dW, \end{cases}$$

If $0 < s < 1$, $q \geq 2$ and $qs > 2$, we see that there is a constant $C > 0$ independent of t and N, such that

$$\mathbb{E}(|(-H)^{\frac{1}{2}} U_N|_{L^2}^2) \leq C.$$

- The embedding $\mathcal{W}^{1,2} \subset L^q$, for any $q < +\infty$ and the fact $\mathcal{W}^{-s,q} \subset \mathcal{W}^{-s',q}$ compact if $s' > s$ deduce:

Theorem (de Bouard-Debussche-F. (2022))

The family of finite dimensional Gibbs measures $(\tilde{\rho_N})_N$ is tight in $\mathcal{W}^{-s,q}$ for any $0 < s < 1$, $q \geq 2$ s.t. $qs > 2$. The weak limit ρ is an invariant measure.