Improved uniform error bounds on time-splitting methods for long-time dynamics of dispersive PDEs

Yue Feng

Laboratoire Jacques-Louis Lions (LJLL), Sorbonne Université

Collaborators: W. Bao (NUS), Y. Cai (BNU), G. Maierhofer (LJLL), K. Schratz (LJLL), W. Yi (HNU), C. Su (THU)

Bridging Classical and Quantum Turbulence, Cargèse, Corsica

Outline

Outline

The nonlinear Klein–Gordon equation

The nonlinear Klein–Gordon equation (NKGE)

$$\begin{cases} \partial_{tt} u(\mathbf{x}, t) - \Delta u(\mathbf{x}, t) + u(\mathbf{x}, t) + \varepsilon^2 u^3(\mathbf{x}, t) = 0, \ \ \mathbf{x} \in \mathbb{T}^d, \ \ t > 0, \\ u(\mathbf{x}, 0) = \phi(\mathbf{x}) = O(1), \quad \partial_t u(\mathbf{x}, 0) = \gamma(\mathbf{x}) = O(1), \quad \mathbf{x} \in \mathbb{T}^d. \end{cases}$$

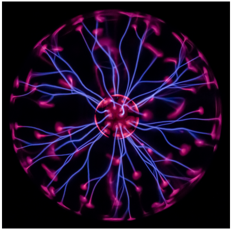
- $u = u(\mathbf{x}, t)$ is real-valued field
- $0 < \varepsilon \leq 1$ is a dimensionless parameter

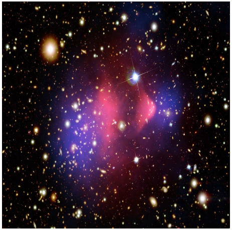
Proposed in 1926 by the physicists Oskar Klein and Walter Gordon

- Describe the motion of spinless particles, e.g. pion
- A relativistic version of nonlinear Schrödinger equation

Applications

- ϕ^4 model with weak coupling constant
- Propagation of dislocations in crystals
- Plasma, e.g. interaction between Langmuir and ion sound waves
- Universe, e.g. dark matter or black-hole evaporation
- Nonlinear optics, nonlinear dynamics of DNA chain, ...





Properties of NKGE

$$\begin{cases} \partial_{tt}u(\mathbf{x},t) - \Delta u(\mathbf{x},t) + u(\mathbf{x},t) + \varepsilon^2 u^3(\mathbf{x},t) = 0, \ \mathbf{x} \in \mathbb{T}^d, \ t > 0, \\ u(\mathbf{x},0) = \phi(\mathbf{x}) = O(1), \quad \partial_t u(\mathbf{x},0) = \gamma(\mathbf{x}) = O(1), \quad \mathbf{x} \in \mathbb{T}^d. \end{cases}$$

- Time symmetric, i.e. unchanged if $t \to -t$
- Hamiltonian (or Energy) conservation

$$\begin{aligned} E(t) &:= \int_{\mathbb{T}^d} \left[|\partial_t u(\mathbf{x},t)|^2 + |\nabla u(\mathbf{x},t)|^2 + |u(\mathbf{x},t)|^2 + \frac{\varepsilon^2}{2}|u(\mathbf{x},t)|^4 \right] d\mathbf{x} \\ &\equiv \int_{\mathbb{T}^d} \left[|\gamma(\mathbf{x})|^2 + |\nabla \phi(\mathbf{x})|^2 + |\phi(\mathbf{x})|^2 + \frac{\varepsilon^2}{2}|\phi(\mathbf{x})|^4 \right] d\mathbf{x} \\ &:= E(0) = O(1), \quad t \geq 0. \end{aligned}$$

- Two different time regimes
  - $O(1)$-time regime, e.g. $0 \leq t \leq T = O(1)$
  - Long-time regime, e.g. $0 \leq t \leq T_\varepsilon = T/\varepsilon^2 = O(\varepsilon^{-2})$

Existing results in $O(1)$-time regime

- Analytical results for Cauchy problem, i.e., the existence, uniqueness and regularity of the solutions: Browder, 62'; Segal, 63; Glassey, 73'; Brenner & von Wahl, 81; Klainerman, 85'; Ginibre & Velo, 85' & 89'; Shatah, 85'; Motai, 89'; Simon & Taflin, 93'; Nakamura & Ozawa, 01', ...
- Numerical methods
  - Finite difference methods: Strauss & Vázquez,78'; Li & Vu-Quoc, 95'; Duncan, 97'; Dehghan, Mohebbi & Asgari, 09', ...
  - Exponential integrators: Hochbruck & Lubich, 99'; Cohen, Hairer & Lubich, 05'; Grimm, 05'; Hochbruck & Ostermann, 10', Bao & Dong, 12'; Gauckler, 15'; Wang, Iserles & Wu, 15'; Baumstark, Faou & Schratz, 18', ...
  - Time-splitting: Mclachlan & Quispel, 03'; Ruf, Bauke & Keitel, 09'; Dong, Xu & Zhao, 14'; Faou & Schratz, 14'; Chartier, Crouseilles, Lemou & Méhats, 15';, ...
  - Spectral methods: Cao & Guo, 93'; Chen, 06', ...

PDE results in long-time regime

$$\begin{cases} \partial_{tt}u(\mathbf{x},t) - \Delta u(\mathbf{x},t) + u(\mathbf{x},t) + \varepsilon^2 u^3(\mathbf{x},t) = 0, \ \mathbf{x} \in \mathbb{T}^d, \ t > 0, \\ u(\mathbf{x},0) = \phi(\mathbf{x}) = O(1), \ \partial_t u(\mathbf{x},0) = \gamma(\mathbf{x}) = O(1), \ \mathbf{x} \in \mathbb{T}^d. \end{cases}$$

$$\Updownarrow w(\mathbf{x},t) = \varepsilon u(\mathbf{x},t)$$

$$\begin{cases} \partial_{tt}w(\mathbf{x},t) - \Delta w(\mathbf{x},t) + w(\mathbf{x},t) + w^3(\mathbf{x},t) = 0, \ \mathbf{x} \in \mathbb{T}^d, \ t > 0, \\ w(\mathbf{x},0) = \varepsilon\phi(\mathbf{x}) = O(\varepsilon), \ \partial_t w(\mathbf{x},0) = \varepsilon\gamma(\mathbf{x}) = O(\varepsilon), \ \mathbf{x} \in \mathbb{T}^d. \end{cases}$$

- Lifespan of the solution on a torus: $O(\varepsilon^{-2})$

  Bourgain, 95'; Ozawa, Tsutaya & Tsutsumi, 96; Delort, 96', 97', 98' & 09'; Keel & Tao, 99'; Sunagawa, 03'; Delort & Szeftel, 04', ...

Questions:

- Error bounds of numerical methods up to time $t = T/\varepsilon^2$
- How the error bounds depend on ε explicitly ?

Outline

Error bounds for FDTD method

$$\begin{cases} \partial_{tt}u(x,t) - \partial_{xx}u(x,t) + u(x,t) + \varepsilon^2 u^3(x,t) = 0, \ x \in (a,b), \ t > 0, \\ u(x,0) = \phi(x), \ \partial_t u(x,0) = \gamma(x), \ x \in \overline{\Omega} = [a,b], \end{cases}$$

- Difference Operators $\delta_t^2 u_j^n = \frac{u_j^{n+1} - 2u_j^n + u_j^{n-1}}{\tau^2}, \delta_x^2 u_j^n = \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{h^2}$
- The Crank-Nicolson finite difference (CNFD) method

$$\delta_t^2 u_j^n - \frac{1}{2}\delta_x^2 \left(u_j^{n+1} + u_j^{n-1}\right) + \frac{1}{2}\left(u_j^{n+1} + u_j^{n-1}\right) + \frac{\varepsilon^2 \left((u_j^{n+1})^4 - (u_j^{n-1})^4\right)}{4\left(u_j^{n+1} - u_j^{n-1}\right)} = 0$$

- Error bounds [1]

$$\|e^n\|_{l^2} + \left\|\delta_x^+ e^n\right\|_{l^2} \lesssim \frac{h^2}{\varepsilon^2} + \frac{\tau^2}{\varepsilon^2}, \quad 0 \le n \le \frac{T/\varepsilon^2}{\tau}$$

- Resolution: $h = O(\varepsilon), \ \tau = O(\varepsilon)$ Under resolution

[1] W. Bao, Y. Feng, W. Yi, **Commun. Comput. Phys.**, 26(5), 1307–1334, 2019.

Error estimates for the FDFS method

- Apply Fourier spectral method in space $\rightarrow$ FDFS method: u_M^n
- Error bounds of the FDFS method up to the time at $O(\varepsilon^{-2})$

$$\|u(x,t_n) - u_M^n(x)\|_{H^1} \lesssim h^m + \frac{\tau^2}{\varepsilon^2}, \quad 0 \le n \le \frac{T/\varepsilon^2}{\tau}$$

- Resolution: $h = O(1)$, $\tau = O(\varepsilon)$ Space: Optimal

Table: Spatial errors of the FDFP method for the NKGE

$e_{h,\tau_e}(t=1/\varepsilon^2)$	$h_0 = \pi/2$	$h_0/2$	$h_0/2^2$	$h_0/2^3$
$\varepsilon_0 = 1$	4.87E-2	1.58E-2	1.30E-4	6.17E-8
$\varepsilon_0/2$	3.43E-2	1.31E-2	1.32E-4	3.03E-7
$\varepsilon_0/2^2$	1.28E-1	3.85E-3	1.28E-5	7.19E-7
$\varepsilon_0/2^3$	8.30E-2	9.65E-3	6.83E-5	5.26E-6

Uniform error bounds for EWI-FP method

Exponential wave integrator Fourier pseudospectral (EWI-FP) method

$$u_j^{n+1} = \sum_{l\in\mathcal{T}_M} \widetilde{u}_l^{n+1} e^{i\mu_l(x_j-a)}, \quad j=0,1,\cdots,M,$$

with

$$\widetilde{(u_M^1)}_l = \cos(\tau\zeta_l)\widetilde{\phi}_l + \frac{\sin(\tau\zeta_l)}{\zeta_l}\widetilde{\gamma}_l + \frac{\varepsilon^2(\cos(\tau\zeta_l)-1)}{\zeta_l^2}\widetilde{f}_l^0(0),$$

$$\widetilde{(u_M^{n+1})}_l = -\widetilde{(u_M^{n-1})}_l + 2\cos(\tau\zeta_l)\widetilde{(u_M^n)}_l + \frac{2\varepsilon^2(\cos(\tau\zeta_l)-1)}{\zeta_l^2}\widetilde{f}_l^n(0),\ n\geq 1.$$

- Uniform error bounds for EWI-FP method [2]

$$\|u(\cdot,t_n)-u^n\|_{H^1} \lesssim h^m + \tau^2, \quad 0\leq n\leq \frac{T/\varepsilon^2}{\tau}$$

- Resolution: $h=O(1),\ \tau=O(1)$ Optimal

[2] Y. Feng, W. Yi, **Multiscale Model. Simul.**, 19(3), 1212–1235, 2021.

Outline

A relativistic NLSE reformulation

$$\begin{cases} \partial_{tt}u(x,t) - \partial_{xx}u(x,t) + u(x,t) + \varepsilon^2 u^3(x,t) = 0, \ x \in \Omega, \ t > 0, \\ u(x,0) = \phi(x), \ \partial_t u(x,0) = \gamma(x), \ x \in \overline{\Omega} = [a,b]. \end{cases}$$

A relativistic NLSE reformulation:

- Define the operator $\langle \nabla \rangle = \sqrt{1-\Delta}$
- Denote $v(x,t) = \partial_t u(x,t)$ and $\psi(x,t) = u(x,t) - i\langle \nabla \rangle^{-1} v(x,t)$
- The NKGE is equivalent to the following relativistic NLSE:

$$\begin{cases} i\partial_t \psi(x,t) + \langle \nabla \rangle \psi(x,t) + \varepsilon^2 \langle \nabla \rangle^{-1} f\big(\frac{1}{2}(\psi + \overline{\psi})\big)(x,t) = 0, \\ \psi(x,0) = \psi_0(x) := \phi(x) - i\langle \nabla \rangle^{-1}\gamma(x), \end{cases}$$

where $f(\psi) = \psi^3$ and $\overline{\psi}$ denotes the complex conjugate of ψ.

Time-splitting method

$$i\partial_t\psi(x,t) + \langle\nabla\rangle\psi(x,t) + \varepsilon^2\langle\nabla\rangle^{-1} f\Big(\frac{1}{2}(\psi+\overline{\psi})\Big)(x,t) = 0$$

- Step 1: linear part

$$\begin{cases} i\partial_t\psi(x,t) + \langle\nabla\rangle\psi(x,t) = 0 \\ \psi(x,0) = \psi_0(x) \end{cases} \implies \psi(\cdot,t) = \varphi_T^t(\psi_0) := e^{it\langle\nabla\rangle}\psi_0.$$

- Step 2: nonlinear part

$$\begin{cases} i\partial_t\psi(x,t) + \varepsilon^2\langle\nabla\rangle^{-1} f\Big(\frac{1}{2}(\psi+\overline{\psi})\Big)(x,t) = 0 \\ \psi(x,0) = \psi_0(x) \end{cases}$$

$$\implies \psi(x,t) = \varphi_V^t(\psi_0) := \psi_0(x) + \varepsilon^2 t F(\psi_0(x)),$$

$$F(\psi) = i\langle\nabla\rangle^{-1}G(\psi), \qquad G(\psi) = f\Big(\frac{1}{2}(\psi+\overline{\psi})\Big).$$

Time-splitting Fourier Pseudospectral (TSFP) method

- The second-order Strang splitting:

$$\psi^{[n+1]} = \mathcal{S}_\tau(\psi^{[n]}) = \varphi_T^{\tau/2} \circ \varphi_V^{\tau} \circ \varphi_T^{\tau/2}(\psi^{[n]}), \quad 0 \le n \le \frac{T/\varepsilon^2}{\tau} - 1,$$

$$\psi^{[0]} = \psi_0 = \phi - i\langle\nabla\rangle^{-1}\gamma.$$

- Time-splitting Fourier Pseudospectral (TSFP) discretization:

$$\psi_j^{(n,1)} = \sum_{l \in \mathcal{T}_M} e^{i\frac{\tau\zeta_l}{2}} \widetilde{(\psi^n)}_l e^{i\mu_l(x_j - a)},$$

$$\psi_j^{(n,2)} = \psi_j^{(n,1)} + \varepsilon^2 \tau F_j^n, \quad F_j^n = i \sum_{l \in \mathcal{T}_M} \frac{1}{\zeta_l} \left(\widetilde{G(\psi^{(n,1)})}\right)_l e^{i\mu_l(x_j - a)}$$

$$\psi_j^{n+1} = \sum_{l \in \mathcal{T}_M} e^{i\frac{\tau\zeta_l}{2}} \left(\widetilde{\psi^{(n,2)}}\right)_l e^{i\mu_l(x_j - a)}.$$

- u_j^n and v_j^n can be recovered by

$$u_j^{n+1} = \frac{1}{2}\left(\psi_j^{n+1} + \overline{\psi_j^{n+1}}\right), \; v_j^{n+1} = \frac{i}{2} \sum_{l \in \mathcal{T}_M} \zeta_l \left[\left(\widetilde{\psi^{n+1}}\right)_l - \left(\widetilde{\overline{\psi^{n+1}}}\right)_l\right] e^{i\mu_l(x_j - a)}.$$

Error estimates for the TSFP method

- Assumptions on the exact solution $u(x,t)$ of the NKGE:

$$\|u\|_{L^\infty\left([0,T_\varepsilon];H_{\mathrm{p}}^{m+1}\right)} \lesssim 1, \quad \|\partial_t u\|_{L^\infty\left([0,T_\varepsilon];H_{\mathrm{p}}^{m}\right)} \lesssim 1, \quad m \geq 1.$$

- Uniform error bounds for the TSFP method [3]

Theorem

There exist $h_0 > 0$ and $\tau_0 > 0$ sufficiently small and independent of ε such that, for any $0 < \varepsilon \leq 1$, when $0 < h \leq h_0$ and $0 < \tau \leq \tau_0$, we have the following error estimates for $0 \leq n \leq \frac{T/\varepsilon^2}{\tau}$,

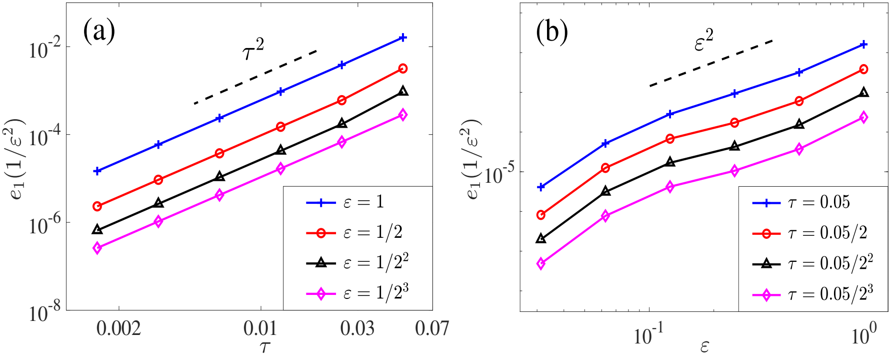
$$\|u(\cdot,t_n) - I_M u^n\|_{H^1} + \|\partial_t u(\cdot,t_n) - I_M v^n\|_{L^2} \lesssim h^m + \tau^2.$$

Moreover, $I_M u^n$ and $I_M v^n$ preserves the regularities, i.e.,

$$I_M u^n \in H_{\mathrm{p}}^{m+1}, \quad I_M v^n \in H_{\mathrm{p}}^{m}.$$

[3] W. Bao, Y. Feng, C. Su, **Math. Comp.**, 91 (334), 811–842, 2022.

Numerical results for the TSFP method



Improved uniform error bounds: $\tau^2 \Longrightarrow \varepsilon^2\tau^2$

Improved uniform error bounds for the TSFP method

Theorem

Under the same assumption of the exact solution, there exist $h_0 > 0$ *and* $0 < \tau_0 < 1$ *sufficiently small and independent of* ε *such that, for any* $0 < \varepsilon \le 1$*, when* $0 < h \le h_0$ *and* $0 < \tau < \alpha \frac{\pi(b-a)\tau_0}{2\sqrt{\tau_0^2(b-a)^2+4\pi^2(1+\tau_0^2)}}$ *for a fixed constant* $\alpha \in (0,1)$*, we have the improved uniform error bounds* [4]

$$\|u(\cdot,t_n) - I_M u^n\|_{H^1} + \|\partial_t u(\cdot,t_n) - I_M v^n\|_{L^2} \lesssim h^m + \varepsilon^2\tau^2 + \tau_0^{m+1}.$$

In particular, if $u, \partial_t u \in H^\infty$*, we could obtain*

$$\|u(\cdot,t_n) - I_M u^n\|_{H^1} + \|\partial_t u(\cdot,t_n) - I_M v^n\|_{L^2} \lesssim h^m + \varepsilon^2\tau^2.$$

[4] W. Bao, Y. Cai, Y. Feng, **SIAM J. Numer. Anal.**, to appear (arXiv: 2109.14902).

Regularity compensation oscillation (RCO)

Tools:

- Multiscale analysis: Regularity compensation oscillation (RCO)
- Twisted variable: $\phi(x,t) = e^{-it\langle\nabla\rangle}\psi(x,t)$

Key idea of RCO:

- Choose a frequency cut-off parameter τ_0
- Control the high frequency modes ($> \frac{1}{\tau_0}$) by the smoothness of the exact solution
- Analyze the low frequency modes ($\leq \frac{1}{\tau_0}$) by the phase cancellation and energy method

Remarks

The complex NKGE with a general power nonlinearity

$$\begin{cases}\partial_{tt}u(\mathbf{x},t) - \Delta u(\mathbf{x},t) + u(\mathbf{x},t) + \varepsilon^{2p}|u(\mathbf{x},t)|^{2p}u(\mathbf{x},t) = 0, \\ u(\mathbf{x},0) = u_0(\mathbf{x}), \quad \partial_t u(\mathbf{x},0) = u_1(\mathbf{x}).\end{cases}$$

- Introduce $\eta_\pm(x,t) = u(x,t) \mp i\ \langle\nabla\rangle^{-1}v(x,t)$
- Denote $f(\varphi) = |\varphi|^{2p}\varphi$, then the complex NKGE can be reformulated into the following coupled relativistic NLSEs:

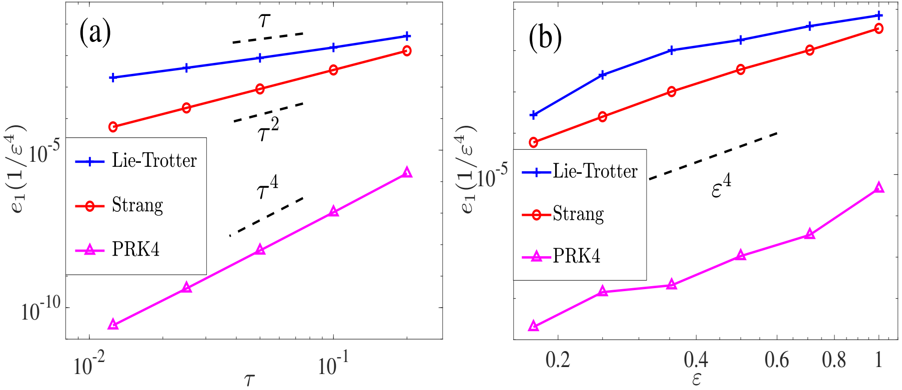
$$\begin{cases} i\partial_t\eta_\pm \pm \langle\nabla\rangle\eta_\pm \pm \varepsilon^{2p}\langle\nabla\rangle^{-1}f\left(\frac{1}{2}\eta_+ + \frac{1}{2}\eta_-\right) = 0, \\ \eta_\pm(t=0) = u_0 \mp i\ \langle\nabla\rangle^{-1}v_0. \end{cases}$$

- Improved uniform error bounds for TSFP method up to $t = \frac{T}{\varepsilon^{2p}}$

$$\|u(\cdot,t_n) - I_N u^n\|_{H^1} + \|\partial_t u(\cdot,t_n) - I_N v^n\|_{L^2} \lesssim h^m + \varepsilon^{2p}\tau^2$$

Numerical results

$$p = 2, t = \frac{1}{\varepsilon^4}$$



Comparisons of different numerical methods

Resolution / Temporal / Spatial	FD	EWI	TS
2nd FD	$h = O(\varepsilon)$ $\tau = O(\varepsilon)$	$h = O(\varepsilon)$ $\tau = O(1)$	$h = O(\varepsilon)$ $\tau = O(1)$
4cFD	$h = O(\varepsilon^{1/2})$ $\tau = O(\varepsilon)$	$h = O(\varepsilon^{1/2})$ $\tau = O(1)$	$h = O(\varepsilon^{1/2})$ $\tau = O(1)$
Spectral	$h = O(1)$ $\tau = O(\varepsilon)$	$h = O(1)$ $\tau = O(1)$	$h = O(1)$ $\tau = O(1)$

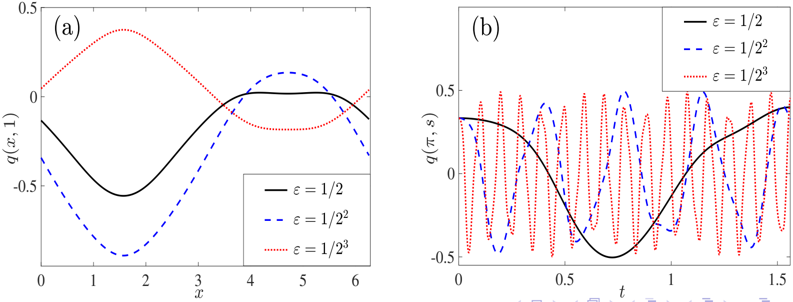
Outline

Extension to an oscillatory NKGE

- Rescaling in time $s = \varepsilon^2 t$, $q(\mathbf{x}, s) := u(\mathbf{x}, s/\varepsilon^2) = u(\mathbf{x}, t)$
- The oscillatory NKGE

$$\begin{cases} \varepsilon^4 \partial_{ss} q(\mathbf{x}, s) - \Delta q(\mathbf{x}, s) + q(\mathbf{x}, s) + \varepsilon^2 q^3(\mathbf{x}, s) = 0, \ \ x \in \mathbb{T}^d, \ \ s > 0, \\ q(\mathbf{x}, 0) = \phi(\mathbf{x}), \quad \partial_s q(\mathbf{x}, 0) = \varepsilon^{-2} \gamma(\mathbf{x}), \quad x \in \mathbb{T}^d. \end{cases}$$

- NKGE with weak nonlinearity at $T/\varepsilon^2 \Longleftrightarrow$ Oscillatory NKGE at T
- Oscillation: wavelength at $O(1)$ in space and $O(\varepsilon^2)$ in time.



Extensions

- (Nonlinear) Schrödinger equation
  Improved uniform error bounds of the time-splitting methods for the long-time (nonlinear) Schrödinger equation, *Math. Comp.*, 92(341), 1109–1139, 2023.
- Dirac equation
  Improved uniform error bounds for the time-splitting method for the long-time dynamics of the Dirac equation with small potentials, *Multiscale Model. Simul.*, 20(3), 1040–1062, 2022.
- Nonlinear Dirac equation
  Improved uniform error bounds for the time-splitting method for the long-time dynamics of the weakly nonlinear Dirac equation, arXiv: 2203.05886.

Sine-Gordon equation

$$\begin{cases} \partial_{tt}u(\mathbf{x},t) - \Delta u(\mathbf{x},t) + \sin(u(\mathbf{x},t)) = 0, & \mathbf{x} \in \Omega, \quad t > 0, \\ u(\mathbf{x},0) = \varepsilon\phi(\mathbf{x}) = O(\varepsilon), \quad \partial_t u(\mathbf{x},0) = \varepsilon\gamma(\mathbf{x}) = O(\varepsilon), & \mathbf{x} \in \Omega. \end{cases}$$

$$\Updownarrow w(\mathbf{x},t) = u(\mathbf{x},t)/\varepsilon$$

$$\begin{cases} \partial_{tt}w(\mathbf{x},t) - \Delta w(\mathbf{x},t) + \frac{1}{\varepsilon}\sin(\varepsilon w(\mathbf{x},t)) = 0, & \mathbf{x} \in \Omega, \quad t > 0, \\ w(\mathbf{x},0) = \phi(\mathbf{x}) = O(1), \quad \partial_t w(\mathbf{x},0) = \gamma(\mathbf{x}) = O(1), & \mathbf{x} \in \Omega. \end{cases}$$

- Rewrite the SGE as

$$\partial_{tt}w(\mathbf{x},t) + \langle\nabla\rangle^2 w(\mathbf{x},t) + \left(\frac{1}{\varepsilon}\sin(\varepsilon w(\mathbf{x},t)) - w(\mathbf{x},t)\right) = 0$$

- Improved uniform error bound for $0 \le n \le \frac{T/\varepsilon^2}{\tau}$ as[5]

$$\|w(\cdot,t_n) - I_M w^n\|_{H^1} + \|\partial_t w(\cdot,t_n) - I_M z^n\|_{L^2} \lesssim h^m + \varepsilon^2\tau$$

[5] Y. Feng, K.Schratz, arXiv:2211.09402.

Cubic Schrödinger equation

$$\begin{cases} i\partial_t u(x,t) = -\partial_x^2 u(x,t) + \varepsilon^2 |u(x,t)|^2 u(x,t), & x \in \mathbb{T},\ t>0, \\ u(x,0) = \phi(x), & x \in \mathbb{T}. \end{cases}$$

- Introduce the twisted variable $v(t) = e^{-it\partial_x^2} w(t)$ satisfying

$$\begin{cases} i\partial_t v(t) = \varepsilon^2 e^{-it\partial_x^2}\left[|e^{it\partial_x^2} v(t)|^2 e^{it\partial_x^2} v(t)\right], & x \in \mathbb{T},\ t>0, \\ v(x,0) = \phi(x), & x \in \mathbb{T}. \end{cases}$$

- Duhamel's formula

$$v(t_n+\tau) = v(t_n) - i\varepsilon^2 \int_0^\tau e^{-i(t_n+s)\partial_x^2}\left[|e^{i(t_n+s)\partial_x^2} v(t_n+s)|^2 e^{i(t_n+s)\partial_x^2} v(t_n+s)\right] ds$$

- Approximating $v(t_n+s) \approx v(t_n)$, the the integral is expressed as

$$I^\tau(v,t_n) = \sum_{l\in\mathbb{Z}} e^{ilx} \sum_{\substack{l_1,l_2,l_3\in\mathbb{Z} \\ l=-l_1+l_2+l_3}} e^{it_n(l^2+l_1^2-l_2^2-l_3^2)} \overline{\hat{v}_{l_1}} \hat{v}_{l_2} \hat{v}_{l_3} \int_0^\tau e^{is(l^2+l_1^2-l_2^2-l_3^2)} ds$$

Low regularity integrators

- Relationship: $l^2 + l_1^2 - l_2^2 - l_3^2 = 2l_1^2 - 2l_1(l_2 + l_3) + 2l_2 l_3$
- For the resonant case $l^2 + l_1^2 - l_2^2 - l_3^2 = 0$,

$$\int_0^\tau e^{is(l^2+l_1^2-l_2^2-l_3^2)} ds = \tau$$

- For the non-resonant cases $l^2 + l_1^2 - l_2^2 - l_3^2 \neq 0$,

$$\int_0^\tau e^{2isl_1^2} ds = \tau\varphi_1(2i\tau l_1^2), \quad \varphi_1(z) = \frac{e^z - 1}{z}$$

- New low regularity integrator[6]

$$v^{n+1} = v^n - i\tau\varepsilon^2 \Big[e^{-it_n\partial_x^2} \Big((e^{it_n\partial_x^2} v^n)^2 (\varphi_1(-2i\tau\partial_x^2) e^{-it_n\partial_x^2} \overline{v^n}) \Big) + 2(\widehat{g(v^n)})_0 v^n - h(v^n) \Big],$$

$$g(v) = v(1 - \varphi_1(-2i\tau\partial_x^2))\overline{v}, \quad (\widehat{h(v)})_l = (1 - \varphi_1(2il^2\tau))\overline{\hat{v}_l}\hat{v}_l\hat{v}_l$$

[6] Y. Feng, G. Maierhofer, K.Schratz, arXiv:2302.00383.

Low regularity integrators

- Improved uniform error bound for $m \geq 2$

$$\|u(t_n) - u^n\|_{H^1} \lesssim \varepsilon^2\tau + \tau_0^{m-1}, \quad 0 \leq n \leq \frac{T/\varepsilon^2}{\tau}$$

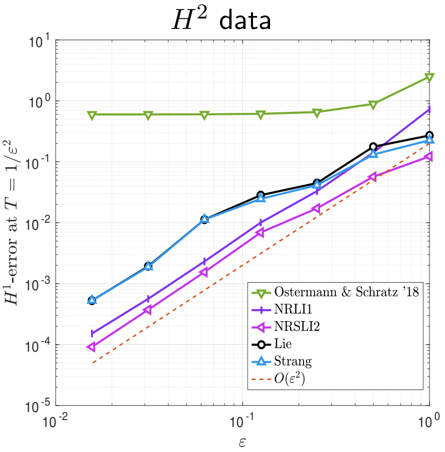
- Self-adjoint map, i.e., $\mathcal{S}_\tau = \mathcal{S}_{-\tau}^{-1}$
- Construct the symmetric method as a composition scheme in the form

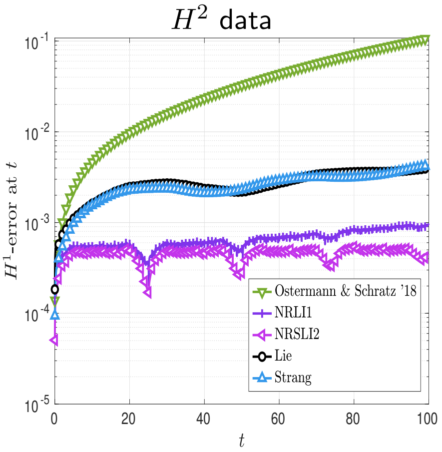
$$u^{n+1} = \mathcal{S}_{-\tau/2}^{-1} \circ \mathcal{S}_{\tau/2}(u^n)$$

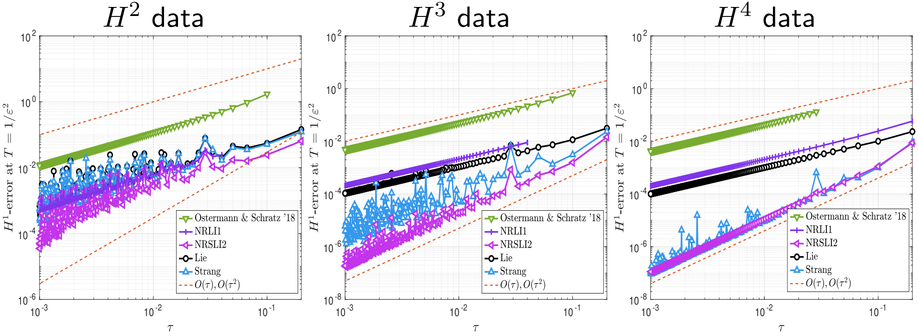
- Symmetric low regularity integrator

$$\begin{aligned} u^{n+1} &= \mathcal{M}_\tau(u^n, u^{n+1}) \\ &:= e^{i\tau\partial_x^2}\left[u^n - \frac{i\tau\varepsilon^2}{2}(u^n)^2(\varphi_1(-i\tau\partial_x^2)\overline{u^n})\right] - \frac{i\varepsilon^2\tau}{2}\left((u^{n+1})^2(\varphi_1(i\tau\partial_x^2)\overline{u^{n+1}})\right) \\ &\quad - \frac{i\varepsilon^2\tau}{2}\left[2(\widehat{g(u^n)})_0 e^{i\tau\partial_x^2}u^n - e^{i\tau\partial_x^2}h(u^n) + 2(\widehat{g(u^{n+1})})_0 u^{n+1} - h(u^{n+1})\right] \end{aligned}$$

Low regularity integrators







H^2 data
H^3 data
H^4 data
H^1-error at $T = 1/\varepsilon^2$
τ
10^{2}
10^{0}
10^{-2}
10^{-4}
10^{-6}
10^{-8}
10^{-3}
10^{-1}
Ostermann & Schratz '18
NRLI1
NRSLI2
Lie
Strang
$O(\tau), O(\tau^2)$

Outline

Conclusions and future work

- Conclusions
  - Error bounds of different numerical methods for the NKGE with weak nonlinearity in the long-time regime up to $t = T/\varepsilon^2 = O(\varepsilon^{-2})$
  - Improved uniform error bounds for the time-splitting methods
  - Extensions to other dispersive PDEs
- Future work
  - Design a numerical scheme using large time steps
  - Long-time dynamics for other equations, e.g. KdV type equations
  - For the whole space problem, $hN = O(1/\varepsilon^2)$

Thank you for your attention !