Irreversibility and singularities

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Navier-Stokes and reversibility

\[ \partial_t u + u \nabla u = -\frac{1}{\rho} \nabla p + \nu \Delta u + F \]

**Creeping Flow**
\[ \nu \Delta u = -F \]
- \( F \rightarrow -F \)
- \( u \rightarrow -u \)

Reversible, Equilibrium

**Euler flow**
\[ E=\text{cte} \]
\[ \partial_t u + u \nabla u = -\frac{1}{\rho} \nabla p \]
- \( t \rightarrow -t \)
- \( u \rightarrow -u \)

Reversible, Equilibrium

https://boingboing.net/2020/10/28/liquids-get-mixed-then-unmixed-due-to-stokes-flow.html

Courtesy J.I. Polenco

**Navier-Stokes is irreversible and non-equilibrium due to Viscosity and forcing**

Entropy production = Dissipation
Non-equilibrium stationary states

\[ \partial_t \frac{u^2}{2} dx^3 = \int F \cdot u \, dx^3 - \nu \int (\nabla u)^2 \, dx^3 \]

Traditional way:

We impose $F$ at given viscosity
We measure energy input needed to maintain $E$ statistically constant

\[ \epsilon_I(\nu) \]
Efficency of NESS of Navier-Stokes

\[ \partial_t \int \frac{u^2}{2} \, dx^3 = \int F \cdot u \, dx^3 - \nu \int (\nabla u)^2 \, dx^3 \]

\[ E \quad \epsilon_I \quad \epsilon_d(\nu) \]

Traditional protocol:
We impose \( F \) at given viscosity
We measure energy input needed to maintain \( E \) statistically constant

\[ \mathcal{E} = \frac{\bar{E} k_I}{F} \]
Dissipation ‘anomaly’ and Onsager’s conjecture

**Obs: dissipation does not depend on viscosity (« spontaneous symmetry breaking »)**

Laminar Prop to $\nu$

Turbulent Indep of $\nu$

"...in three dimensions a mechanism for complete dissipation of all kinetic energy, even without the aid of viscosity, is available."

L. Onsager, 1949

*See Eyink&Sreenivasan (2006)*

$$\frac{1}{2} \frac{\partial u}{\partial t} u^2 + \text{div} \left( u \left( \frac{1}{2} u^2 + p \right) - \nu \nabla u \right) = D(u) - \nu (\nabla u)^2$$


If $h > 1/3$ $\Rightarrow$ Euler equation conserves energy, Dissipation in Navier-Stokes by viscosity. *(Eyink 1994, Constantin et al, 1994)*

If $h \leq 1/3$ $\Rightarrow$ Dissipation through irregularities (singularities) Without viscosity! *(Isett, 2018)*

**Regular Test function of width $\ell$**

Inertial dissipation= limit of local energy transfers

\[
D(u) = \lim_{\ell \to 0} \frac{1}{4} \int_{r \leq \ell} d^3r \ \nabla \phi_\ell (r) \cdot \delta u_r |\delta u_r|^2
\]

\[
\delta u(\ell) \sim \ell^h \quad \text{In the limit of } \ell \approx 0
\]

\[
D(u)[x] \propto \lim_{\ell \to 0} \ell^{3h-1}
\]

$\delta u = u(x+r) - u(x)$ Velocity increment

Onsager conjecture links irreversibility and singularity!
In the limit of $u(\ell) \approx h$,

$$D(u) = \lim_{\ell \to 0} \frac{1}{4} \int_{r \leq \ell} d^3 r \ \nabla \phi_\ell(r) \cdot \delta u_r |\delta u_r|^2$$

$$\delta u(\ell) \sim \ell^h$$

In the limit of $\ell \approx 0$

$$D(u)[x] \propto \lim_{\ell \to 0} \ell^{3h-1}$$

Scalar regularity indicator $D_{\ell}^I$

We can use this quantity as an indicator pointing towards less regular regions

Direct investigation using local dissipation

Local Lagrangian and Eulerian velocity measurements

B. Dubrulle, JFM perspectives, 2019
Direct investigation using local dissipation

Courtesy A. Harekrishnan
Dissipation ‘anomaly” and Onsager’s conjecture

**Obs: dissipation does not depend on viscosity (« spontaneous symmetry breaking »)**

Is there anyway we can capture quasi-singularities and weak solutions?

Inertial dissipation: limit of local energy transfers

\[
D(u) = \lim_{\ell \to 0} \frac{1}{4} \int_{r \leq \ell} d^3r \ \nabla \phi_{\ell}(r) \cdot \delta u_r |\delta u_r|^2
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\delta u(\ell) \sim \ell^h \quad \text{In the limit of } \ \ell \approx 0
\]

\[
D(u)[x] \propto \lim_{\ell \to 0} \ell^{3h-1}
\]

\[
\delta u = u(x+r)-u(x) \quad \text{Velocity increment}
\]

If \( h > 1/3 \) \( \Rightarrow \) Euler equation conserves energy, Dissipation in Navier-Stokes by viscosity. *(Eyink 1994, Constantin et al, 1994)*

If \( h \leq 1/3 \) \( \Rightarrow \) Dissipation through irregularities (singularities) Without viscosity! *(Isett, 2018)*

Onsager conjecture links irreversibility and singularity!
Can we just run an Euler/ NSE equation using DNS?

Due to thermalization, any truncated Euler equation ends in equilibrium state with $k^2$ spectrum

Alexakis & Brachet, 2019

« Thermalization » also happens for Navier-Stokes
From DNS to log-lattices

\[ \partial_t \hat{u}_i = P_{ij} \left( -ik_q \hat{u}_q \ast \hat{u}_j + \hat{f}_j \right) - \nu_r k^2 \hat{u}_i, \]

**Fourier grid**

\[ \hat{u} \ast \hat{u} \quad m = n + q, \ (m,n,q) \in \mathbb{Z}^3 \]

**Log grid**

\[ \lambda^m = \lambda^n + \lambda^q, \ (m,n,q) \in \mathbb{Z}^3 \]

\[ \lambda = 2 \quad (z = 3^D). \]
\[ \lambda = \sigma \approx 1.325 \quad (z = 12^D) \]
\[ \lambda = \Phi \approx 1.618 \quad (z = 6^D) \]
\[ 1 = \lambda^b - \lambda^a, \ 0 < a < b \]

Campolina & Mailybaev, 2018
LL-Euler, adaptative grid, Equilibrium reversible

Non-dissipative Singularity!  
Self-similar blow up in finite time

$E(k) = k^{-7/3}$  
($h = 2/3$)

Campolina & Mailybaev, 2018
A new protocol to capture weak solutions

Traditional way:

$$\partial_t \int \frac{u^2}{2} \, dx^3 = \int F \cdot u \, dx^3 - \nu \int (\nabla u)^2 \, dx^3 + \int D(u) \, d^3 x$$

We impose $F$ at given viscosity
We measure energy input needed to maintain $E$ statistically constant

Alternative way using RNS

We impose $F$ at given $E$
We measure the viscosity needed to sustain constant $E$

$$\frac{\partial u}{\partial t} + (u \cdot \nabla) u = -\nabla p + \nu_r \nabla^2 u + f,$$

Control parameter:

$$R_r = \frac{1}{\mathcal{E}} = \frac{F}{E k_I}$$

Gallavotti, PRL, 1996

If No anomaly

$$\nu_r[u] = \frac{\int_D f \cdot u \, dx}{\int_D (\nabla \times u)^2 \, dx}.$$
Navier-Stokes and reversibility

\[ \partial_t u + u \nabla u = -\frac{1}{\rho} \nabla p + \nu \Delta u + F \]

- **Creeping Flow**
  - Reversible Equilibrium
  - Reversible NS
    - Reversible
    - Non-Equilibrium
    - Euler flow

- **Euler flow**
  - Reversible Equilibrium
    - Reversible
    - Non-Equilibrium

*Courtesy G. Krstulovic*
Log-Lattice reversible Navier-Stokes

\[ K_{\text{max}} = 270 \]
\[ K_{\text{max}} = 1768 \]
\[ K_{\text{max}} = 75,959 \]

\[ \langle \nu_r \rangle \propto \mathcal{R}_r^{2.2} \]

Costa et al., 2023
Log-Lattice reversible Navier-Stokes

$E(k) \propto k^{-1.5}$

$\langle \nu_r \rangle \propto k^2$

Costa et al., 2023
DNS reversible Navier-Stokes

Costa et al., 2023
What is happening in the transition zone?

Costa et al, 2023
There is a second order phase transition following mean field exponents.
Is there anomalous dissipation?

No anomalous dissipation

\[ \nu \sim (R_r - 2.5)^3 \]

Costa et al., 2023
What is the limit of infinite resolution?

Euler Dissipative weak solution

\[ \nu_r = \frac{\langle f \cdot u \rangle - D(u)}{\langle \omega^2 \rangle} \]

\[ \nu \sim (R_r - 2.5)^3 \]

Viscous no-dissipative Hydrodynamic branche

Costa et al., 2023
What is the limit of infinite resolution?

\[ \nu_r = \frac{\langle f \cdot u \rangle - D(u)}{\langle \omega^2 \rangle} \]

\[ \nu \sim (R_r - 2.5)^3 \]

Costa et al., 2023
Conjecture

Can we observe these weak solutions?

Construction via convex integration
Matsumoto&Frisch
Buckmaster et al....

\[ \nu \sim (R_r - 2.5)^3 \]
General case with adaptive grid

\[ \alpha = 2h - 1 = \frac{7}{3} \]

\[ R_r = 0 \]

Non-dissipative Singularity!

( \( h < \frac{1}{3} \))

Blow-up with a different exponent

Costa et al, 2023
What is happening when $k_{\text{max}}$ is reached?

**DNS**

- $E(k)$ vs $k$
  - $k^{-5/3}$
  - $k^{-7/3}$

**LogLattice**

- $E(k)$ vs $k$
  - $k^{-5/3}$
  - $k^{-7/3}$

**ω_{max}(t)** vs $t$

Fehn et al. JFM, 2022

Barral et al., 2023
Conjecture (2): scenario to obtain weak solution

Anomalous spectral laws in differential models of turbulence

Thalabard et al, 2015

Figure 4. Left: Compensated MHD spectra obtained numerically from the second-order differential model at different times: earlier than $t_*$ (in blue) and later than $t_*$ (in red). The spectra are normalized by the total energy. The times are reported by the dots on the right inset. The blue dots correspond to $t/t_* = -1 \approx -6 \cdot 10^{-1}, -4 \cdot 10^{-1}, -10^{-1}, -4 \cdot 10^{-2}, -10^{-2}, -2 \cdot 10^{-3}, 0$; the red dots to $t/t_* = 1 \approx 4 \cdot 10^{-3}, 4 \cdot 10^{-2}, -10^{-1}, 4 \cdot 10^{-1}, 7 \cdot 10^{-2}$. The right inset shows the time evolution of the energy. $t_* \approx 8.3 \cdot 10^{-3}$ marks the onset of energy dissipation. The figure is made using the data obtained by E.Buchlin and previously reported in [23] (in a different form).

Forward self-similar Blow-up in finite time $k^{-\alpha}$ (h>1/3, not dissipative)  \[ \text{Backward self-similar front resulting in } k^{-\beta} \text{ (h<1/3, dissipative)} \]
Bridging classical and quantum turbulence
Observations

GP Equations

\[ i\varepsilon \psi_t = -\frac{\varepsilon^2}{2} \Delta \psi + h(|\psi|^2)\psi, \quad x \in \mathbb{R}^d, \]
\[ \psi(\cdot, 0) = \psi_I(\cdot) \]

Madelung

Quantum Hydrodynamic equations

\[ \rho_t + \text{div}(\rho u) = 0, \quad t > 0, \; x \in \Omega \subseteq \mathbb{R}^d, \]
\[ (\rho u)_t + \text{div}(\rho u \otimes u) + \nabla P(\rho) = \frac{\varepsilon^2}{2} \rho \nabla \left( \frac{\Delta \sqrt{\rho}}{\sqrt{\rho}} \right), \]

\textit{Blow-up in finite time with boundaries}

ON THE BLOWING UP OF SOLUTIONS TO QUANTUM HYDRODYNAMIC MODELS ON BOUNDED DOMAINS

IRENE M. GAMBA\textsuperscript{†}, MARIA PIA GUALDANI\textsuperscript{‡}, AND PING ZHANG\textsuperscript{§}

Abstract. The blow-up in finite time for the solutions to the initial-boundary value problem associated to the multi-dimensional quantum hydrodynamic model in a bounded domain is proved. The model consists on conservation of mass equation and a momentum balance equation equivalent to a compressible Euler equations corrected by a dispersion term of the third order in the momentum balance. The proof is based on a-priori estimates for the energy functional for a new observable constructed with an auxiliary function, and it is shown that, under suitable boundary conditions and assumptions on the initial data, the solution blows up after a finite time.
A R-quantum hydro model as a toy model for SF?

\[
\begin{align*}
\rho_t + \text{div}(\rho u) &= 0, \quad t > 0, \ x \in \Omega \subseteq \mathbb{R}^d, \\
(\rho u)_t + \text{div}(\rho u \otimes u) + \nabla P(\rho) &= \frac{\varepsilon^2}{2} \rho \nabla \left( \frac{\Delta \sqrt{\rho}}{\sqrt{\rho}} \right) + \nu_r \Delta u + F
\end{align*}
\]

**Advantage:**
*We naturally have the phase transition*
*Two fluid phase: « superfluid »= weak solution; « normal » = viscous solution
« sound »=tygers; coupling automatic between the two phases*
*We can have finite dissipation by weak solutions like observed in superfluids*
*Forcing is easy*
Finite resolution

Costa et al., 2023
Infinite resolution

\[ \nu_r \approx (R_r - 2.5)^3 \]

Superfluid branch
Euler Dissipative weak solution

\[ \nu_r = \frac{< f \cdot u > - D(u)}{< \omega^2 >} \]

Two-fluid Mixture

Normal Fluid Hydrodynamic branch

\[ R_{rc} = 2.5 \]
Using log-lattices, we have identified in RNS a second order transition as a function of efficiency where the fluid changes:
from **viscous Navier-Stokes** flow to **thermalized Euler Flow**

In the infinite resolution limit, the thermalized Euler Flow could tend to **dissipative weak solutions of Euler equation** of finite energy.

- Can we get the possible weak dissipative solutions of LL-RNS?
- Are these things true for DNS-RNS?
- Is RQNS a good model of superfluid? What kind of structures are observed during the blow-up phase and normal fluid building?(role of quantum pressure)