HVBK two-fluid model for quantum turbulence: energy cascade and higher-order statistics

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Outline

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   - The mutual friction force
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   - The internal intermittency
   - The intermittency of small scales
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HVBK two-fluid model

Helium II two-fluid model (Tisza & Landau 1941)

- Normal fluid: ρ^n, v_n, ν^n, carries entropy
- Superfluid: ρ^s, v_s, free of entropy
- Below T_λ mixture of two fluid $\rho = \rho^n + \rho^s$, for $T > T_\lambda$, $\rho^s/\rho = 0$; for $T = 0K$, $\rho^n/\rho = 0$

The mutual friction force

$$\boldsymbol{F}_M = -B\frac{\rho^s\rho^n}{\rho}\frac{\boldsymbol{\omega}\wedge(\boldsymbol{\omega}\wedge(\boldsymbol{u}_s-\boldsymbol{u}_n))}{|\boldsymbol{\omega}|} - B'\frac{\rho^s\rho^n}{\rho}\boldsymbol{\omega}\wedge(\boldsymbol{u}_s-\boldsymbol{u}_n) \tag{1}$$

B and $B^{'}$: two temperature-dependent parameters, $\omega = \nabla\times\boldsymbol{u}_s$,
$\boldsymbol{u}_n$: the normal fluid velocity, $\boldsymbol{u}_s$: the superfluid velocity.

The governing equations

$$\nabla.\boldsymbol{u}_n = 0,\ \nabla.\boldsymbol{u}_s = 0 \tag{2}$$

$$\frac{\partial \boldsymbol{u}_n}{\partial t} + \nabla.(\boldsymbol{u}_n\otimes\boldsymbol{u}_n) = -\nabla p_n + \frac{1}{\rho^n}\boldsymbol{F}_M + \nu^n\Delta\boldsymbol{u}_n + \boldsymbol{f}_{ext} \tag{3}$$

$$\frac{\partial \boldsymbol{u}_s}{\partial t} + \nabla.(\boldsymbol{u}_s\otimes\boldsymbol{u}_s) = -\nabla p_s - \frac{1}{\rho^s}\boldsymbol{F}_M + \nu^s\Delta\boldsymbol{u}_s + \boldsymbol{f}_{ext} \tag{4}$$

HVBK in turbulence

Truncated HVBK $\nu^s/\nu^n = cst \ll 1$

- LES: ① Merahi et al. [2006] ②Bakhtaoui and Merahi [2015]
- Shell model: ③ Shukla and Pandit [2016]
- DNS: ④ Roche et al. [2009]⑤ Salort et al. [2010] ⑥Salort et al. [2012]

Gradually damped HVBK ν^n/ν^s(T)

- Shell model: ⓐ Boué et al. [2013] ⓑBoué et al. [2015]
- DNS: ⓒ Biferale et al. [2018]

Some common observations of previous studies of the HVBK model

- $-5/3$ spectrum for large scales
- Locking of $\boldsymbol{u}_s$ and $\boldsymbol{u}_n$
- Scaling law of structure functions $S_\gamma(r) \propto r^{\gamma/3}$ ⑥ⓒ
- Intermittency for superfluid turbulence (view over different ranges of scales)

Incompatibilities of previous studies of the HVBK model

- T $\downarrow$ leads to $\eta^n \downarrow$, $\eta^s \downarrow$ ③
- T $\downarrow$ leads to $\eta^n \uparrow$, $\eta^s \downarrow$ ⑤⑥
- T $\downarrow$ leads to $\eta^n \downarrow$, $\eta^s \uparrow$ ⓒ
- Enhancement of turbulence intermittency ⓐⓒ
- Depletion of intermittency ③

Objectives of the present research

Mutual friction force

• Statistics of the mutual friction force
• Energy exchange mechanism between each fluid component. Disentanglement of the inner energy exchange

Scaling laws.. if any, and related issues

• Transport equations for the second-order structure functions
• The third-order structure function transfer equation $\rightarrow$ flatness factor

Small-scale intermittency

• Statistics of velocity increments at small scales

The numerical simulations

The 3dt-HVBK solver

A solver of 3D box periodic boundary condition HIT using spectral method (p3dfft, pencil 2D decomposition), large eddy forcing, standard de-aliasing, two-fluid HVBK model.

The truncated HVBK two-fluid model

- $\nu^n/\nu^s = 10$ with $\nu^n = cst$ (independent of temperature)
- Simplified mutual friction expression, with $B = 1.5$ invariant to T

$$\boldsymbol{F}_M = \frac{B}{2}\frac{\rho^s\rho^n}{\rho}|\omega|.(\boldsymbol{u}_s - \boldsymbol{u}_n)$$

- Unique variable $\rho^n/\rho^s = 10,\ 1,\ 0.1,$ for high, intermediate and low temperature.
- External forcing f^n_{ext} and f^s_{ext} with constant energy injection rate ε^*.

Turbulence parameters

- The large scales are invariant with temperature (u', L)
- The small scales are temperature-dependent (λ, η)
- The energy flows averagely from the superfluid to the normal fluid and reaches a balance.

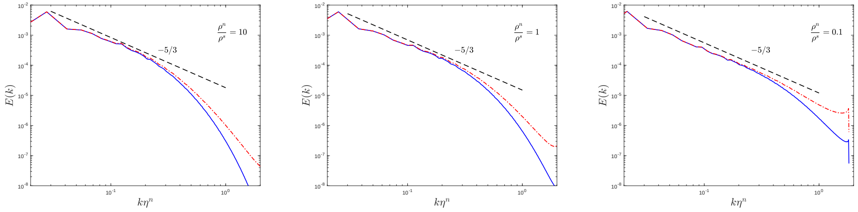
	ρ^n/ρ^s	Re_L	Re_λ	L/η	η/Δ	$\varepsilon/\varepsilon^*$	ϕ/ε^*
Normal fluid	10	580.74	90.37	60.10	1.12	1.06	0.07
	1	570.78	74.09	65.49	1.02	1.58	0.59
	0.1	568.68	54.12	76.00	0.87	3.01	2.03
Superfluid	10	5.78e+03	783.29	202.46	0.33	0.15	-0.85
	1	5.65e+03	600.84	227.32	0.28	0.26	-0.83
	0.1	5.59e+03	369.07	286.42	0.22	0.70	-0.30

Table: $Re_L = \frac{L.u'}{\nu}$, $Re_\lambda = \frac{\lambda.u'}{\nu}$ with $\lambda = \sqrt{15\nu(u')^2/\varepsilon}$, ε is the average energy dissipation rate, $L = ((u')^2)^{3/2}/\varepsilon$ is the scale of the large eddies, η is the Kolmogorov length scales, Δ is the mesh size $\varepsilon^* = \overline{U_i^s f_i^s} = \overline{U_i^n f_i^n}$ is is the energy injection rate at the large scales. $\phi^n = \frac{\rho^s}{\rho}\overline{U_i^n F_i^{ns}}$ is the average energy flux exerted by the friction force on the normal fluid, $\phi^s = -\frac{\rho^n}{\rho}\overline{u_i^s F_i^{ns}}$ is the average energy flux applied by the friction force on the superfluid.

The turbulence statistics

The energy spectrum

- -5/3 in the inertial range
- When temperature decreases: the inertial range extends for both the normal fluid and the superfluid.
- When temperature decreases: the Kolmogorov scales η decrease monotonously.



(a) (b) (c)

Figure: The kinetic energy spectrum for different temperature (a) $\rho^n/\rho^s = 10$, (b) $\rho^n/\rho^s = 1$, (c) $\rho^n/\rho^s = 0.1$. (–) is the normal fluid, ($-\cdot$) is the superfluid, (- -) marks the $-5/3$ slope. Wavenumbers are normalised with the normal fluid Kolmogorov scales, $1/\eta^n$.

The mutual friction force

The energy exchange through the mutual friction

- Knowing that the high enstrophy area are the same, the strong energy exchange only occurs at the strong enstrophy area

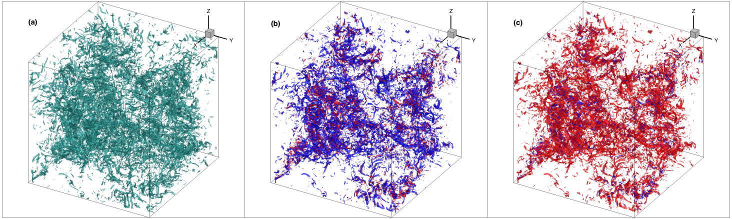
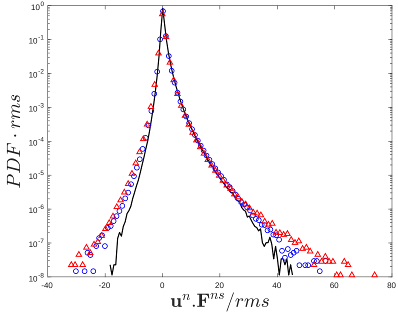


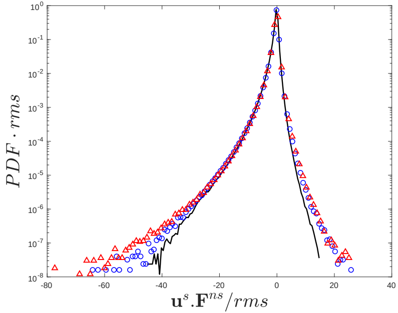
Figure: (a) The snapshot of iso-contour $|\omega_s| = 7\sqrt{\langle|\omega_s|^2\rangle}$, (b) the snapshot of the iso-contour of $u_n.F_{ns}$. (blue) energy gain $5\sqrt{\langle(u_n.F_{ns})^2\rangle}$, (red) energy loss $-3\sqrt{\langle(u_n.F_{ns})^2\rangle}$, (c) the snapshot of the iso-contour of $u_s.F_{ns}$. (blue) energy gain $5\sqrt{\langle(u_s.F_{ns})^2\rangle}$, (red) energy loss $-3\sqrt{\langle(u_s.F_{ns})^2\rangle}$,

The energy exchange through the mutual friction

- Skewed PDF form)
- Stretched tails of the PDF (a source term looks like the dissipation rate)



(a) The ϕ^n in the normal fluid



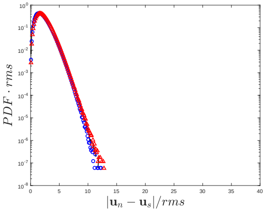
(b) The ϕ^s in the superfluid

Figure: The PDFs of the energy exchange flux normalised by their RMS values for different temperatures (–) $\rho^n/\rho^s = 10$, (○) $\rho^n/\rho^s = 1$, (△) $\rho^n/\rho^s = 0.1$.

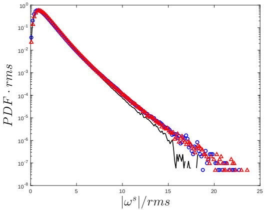
The mutual friction force

The energy exchange through the mutual friction

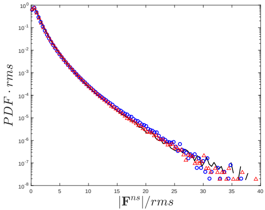
- Non-Gaussian PDF of the mutual friction
- Stretched tails mainly from the enstrophy
- Skewness from the relative velocity



(a) The magnitude of relative velocity



(b) The enstrophy of the superfluid



(c) The magnitude of mutual friction

Figure: The PDFs of the involved variables in the mutual friction force normalised by their RMS values, for different temperature ($-$) $\rho^n/\rho^s = 10$, ($\circ$) $\rho^n/\rho^s = 1$, ($\triangle$) $\rho^n/\rho^s = 0.1$.

The scale-by-scale energy budget

We follow the classical approach in HIT for the second–order structure function transport equations, and integration from 0 to r leads to the 4/3 law, viz.

$$-\overline{\delta u_{\parallel}^{n}(\delta u_i^{n})^2} - \frac{1}{r^2}\int_0^r s^2 \mathcal{L}^n ds + 2\nu^n \frac{d}{dr}\overline{(\delta u_i^n)^2} = \frac{4}{3}\overline{\epsilon}^n r, \tag{5}$$

$$-\overline{\delta u_{\parallel}^{s}(\delta u_i^{s})^2} - \frac{1}{r^2}\int_0^r s^2 \mathcal{L}^s ds + 2\nu^s \frac{d}{dr}\overline{(\delta u_i^s)^2} = \frac{4}{3}\overline{\epsilon}^s r, \tag{6}$$

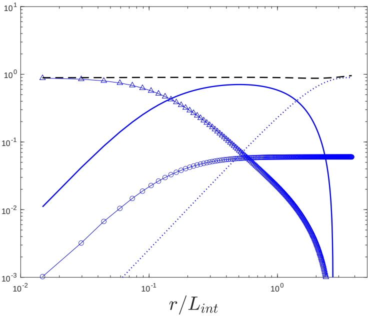
with

$$\mathcal{L}^n \equiv -2\frac{\rho^s}{\rho}\overline{(\delta u_i^n)(\delta F_i^{ns})} - 2\overline{(\delta u_i^n)(\delta f_i^n)}, \tag{7}$$

$$\mathcal{L}^s \equiv 2\frac{\rho^n}{\rho}\overline{(\delta u_i^s)(\delta F_i^{ns})} - 2\overline{(\delta u_i^s)(\delta f_i^s)}, \tag{8}$$

where $u_{\parallel}$ is the velocity component parallel to vector $\vec{r}$, $r = |\vec{r}|$, $\delta f = f(\vec{x}+\vec{r}) - f(\vec{x})$ and for isotropic turbulence δf depends only on r.

$$\underbrace{-\overline{\delta u_{\parallel}(\delta u_i)^2}}_{S3_{ii}} + LF + L_{ext} + \underbrace{2\nu \frac{d}{dr}\overline{(\delta u_i)^2}}_{dS2_{ii}} = \frac{4}{3}\varepsilon r \quad (9)$$



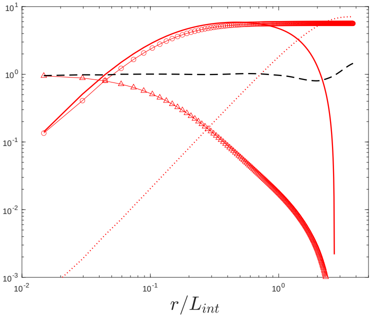


Figure: The case $\rho^n/\rho^s = 10$. (—-) is the convection term $-S3_{ii}^{n,s}$, (○) the mutual friction term $LF^{n,s}$, ($-\triangle$) the viscous term $dS2_{ii}^{n,s}$, ($\cdots$) the external forcing term $L_{ext}^{n,s}$. All terms are normalised by $4/3r\varepsilon^{n,s}$, ($--$) marks the sum of all the terms. (left column) for the normal fluid, (right column) for the superfluid, L_{int} is the integral scale for the total fluid.

$$\underbrace{-\overline{\delta u_{\parallel}(\delta u_i)^2}}_{S3_{ii}} + LF + L_{ext} + \underbrace{2\nu \frac{d}{dr}\overline{(\delta u_i)^2}}_{dS2_{ii}} = \frac{4}{3}\varepsilon r \tag{10}$$

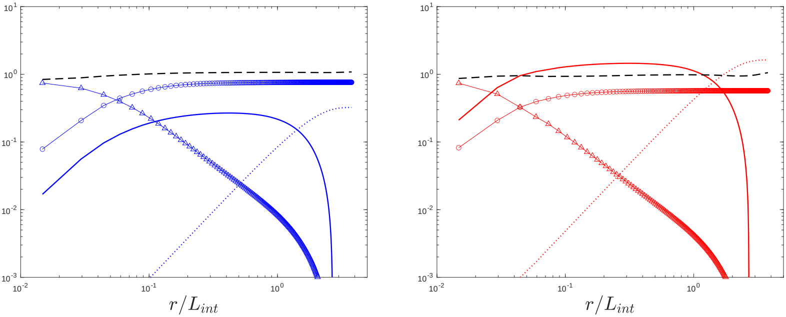


Figure: The case $\rho^n/\rho^s = 0.1$. (—-) is the convection term $-S3_{ii}^{n,s}$, (∘) the mutual friction term $LF^{n,s}$, ($-\triangle$) the viscous term $dS2_{ii}^{n,s}$, ($\cdots$) the external forcing term $L_{ext}^{n,s}$. All terms are normalised by $4/3r\varepsilon^{n,s}$, ($--$) marks the sum of all the terms. (left column) for the normal fluid, (right column) for the superfluid, L_{int} is the integral scale of the total fluid.

$$\partial_t D_{111} + \left(\partial_r + \frac{2}{r}\right) D_{1111} - \frac{6}{r} D_{1122} = -T_{111} + 2\nu C - 2\nu Z_{111} + MF + Fext, \tag{11}$$

with $\partial_r \equiv \partial/\partial r$,

$$\begin{aligned} D_{111} &= \overline{(\delta u)^3}; \\ D_{1111} &= \overline{(\delta u)^4}; \\ D_{1122} &= \overline{(\delta u)^2 (\delta v)^2}; \\ C(r,t) &= -\frac{4}{r^2} D_{111}(r,t) + \frac{4}{r} \partial_r D_{111} + \partial_r \partial_r D_{111}; \\ Z_{111} &= \overline{3\delta u \left[\left(\frac{\partial u}{\partial x_l}\right)^2 + \left(\frac{\partial u'}{\partial x'_l}\right)^2 \right]}, \end{aligned} \tag{12}$$

where $\delta u = u(x+r) - u(x)$ is the longitudinal velocity increment, $\delta v = v(x+r) - v(x)$ is the transverse velocity increment and double indices indicate summation and a prime denotes variables at point $x + r$. Finally,

$$T_{111} = 3\overline{(\delta u)^2 \, \delta \left(\frac{\partial p}{\partial x}\right)}. \tag{13}$$

$$\partial_t D_{111} + \left(\partial_r + \frac{2}{r}\right) D_{1111} - \frac{6}{r} D_{1122} = -T_{111} + 2\nu C - 2\nu Z_{111} + MF + Fext, \qquad (14)$$

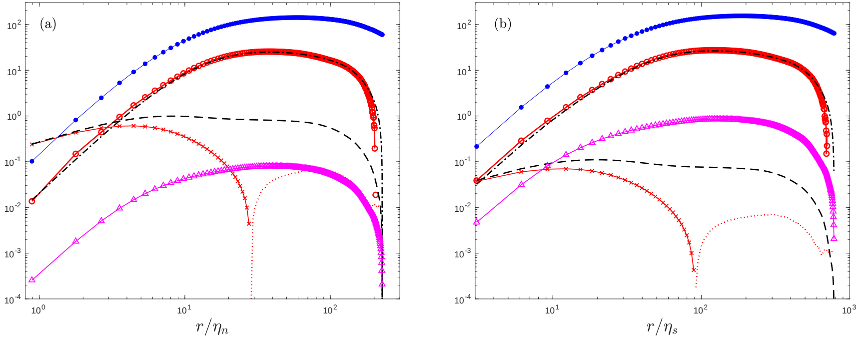


Figure: Balances of different terms in equations for the normal fluid (left) and the superfluid (right) for density ratios $\rho_n/\rho = 0.91$. ($\bullet-$) $(\partial_r + 2/r)D1111$, ($\circ-$) $(\partial_r + 2/r)D1111 - 6/r.D1122$, (black $-\cdot$) $-T111$, ($\times-$) $-2\nu C$ ($\cdot\cdot\cdot$) positive part of $2\nu C$, (- -) $-2\nu Z111$, ($\triangle-$) coupling terms. The plots are dimensionless.

Internal intermittency

$$\partial_t D_{111} + \left(\partial_r + \frac{2}{r}\right) D_{1111} - \frac{6}{r} D_{1122} = -T_{111} + 2\nu C - 2\nu Z_{111} + MF + Fext, \qquad (15)$$

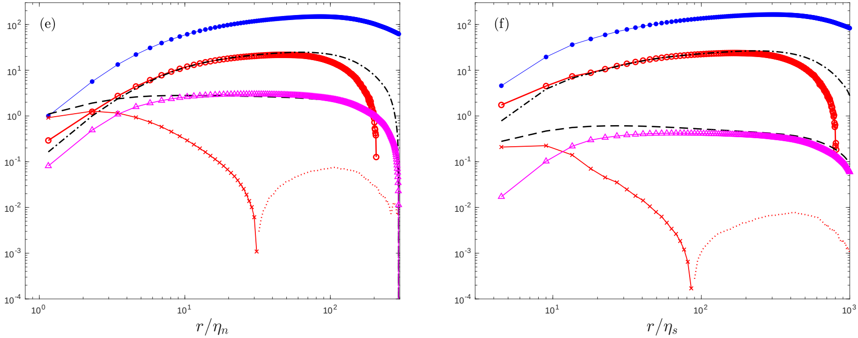


Figure: Balances of different terms in equations for the normal fluid (left) and the superfluid (right) for density ratios $\rho_n/\rho = 0.09$. ($\bullet-$) $(\partial_r + 2/r)D1111$, ($\circ-$) $(\partial_r + 2/r)D1111 - 6/r.D1122$, (black $-\cdot$) $-T111$, ($\times-$) $-2\nu C$ ($\cdot\cdot\cdot$) positive part of $2\nu C$, (- -) $-2\nu Z111$, ($\triangle-$) coupling terms. The plots are dimensionless.

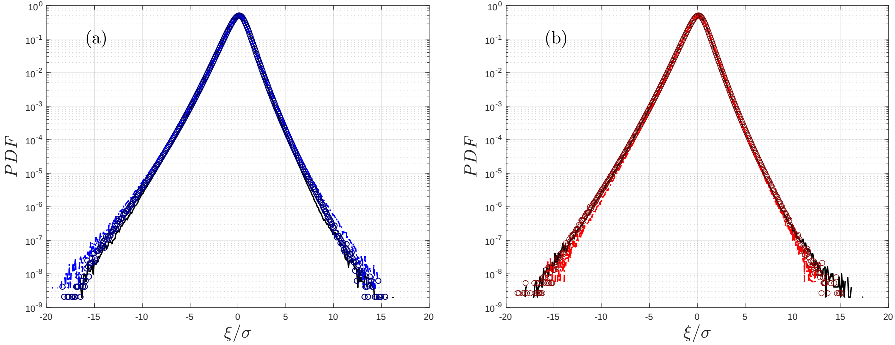


Figure: PDFs of gradients of longitudinal velocity in the normal fluid (a) and superfluid (b). Results for three density ratios: (- ·) $\rho_n/\rho = 0.09$, (○) $\rho_n/\rho = 0.5$, (–) $\rho_n/\rho = 0.91$.

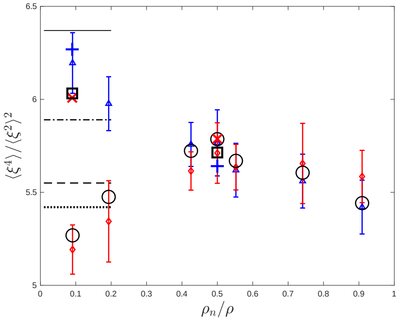


Figure: Flatness factors of the longitudinal velocity gradient $\xi = \partial_x u$ versus density ratio ρ_n/ρ for the normal fluid ($\triangle$), the superfluid ($\diamond$) and total fluid ($\bigcirc$). Error bars are the root-mean-square value of the variance of the flatness factors computed with 20 to 50 snapshots, and 10^8 data points for each snapshot. Horizontal lines mark the flatness factor computed from DNS of classical turbulence (Ishihara, 2007): (· ·) $Re_\lambda = 94.6$, (- -) $R_\lambda = 94.4$, (- ·) $R_\lambda = 167$. (–) $R_\lambda = 173$. All points are computed for $N = 512$, except the following ones, based on $N = 1024$ resolution: big blue $+$ (normal fluid), big red $\times$ (superfluid) and big black $\square$ (total fluid).

Conclusions

A picture of the energy exchange

• The mutual friction exchanges momentum between the two fluid components, averagely it extracts energy from the superfluid and adds energy to the normal fluid.
• The largest energy exchange mainly occurs at regions with strongest vorticity, which is an intermittent effect.

The scale-by-scale energy budget

• Although the mutual friction has a very important impact on the energy balance, the way of energy cascading in the inertial sub-range is unchanged (classical way).
• The local dissipation rate is balanced by both the large-eddy energy injection, **and** the energy exchange through the mutual friction (mainly at small scales).

The intermittency

• The mutual friction shows no effect on internal intermittency across the inertial range
• The energy source term due to the mutual friction modifies the small scale intermittency

Publications

• Z. Zhang, I. Danaila, E. Lévèque and L. Danaila, ”Higher-order statistics and intermittency of a two-fluid HVBK quantum turbulent flow”, J. Fluid Mech., 2023.
• Z. Zhang, M. Gauding, J. Polanco, I. Danaila, E. Leveque and L. Danaila, ”On the energy cascade in HVBK flow”, to be submitted.

Some perspectives

An alternative sub-grid model

Merahi et al. [2006] claims that the sub-grid model of type Smagorinsky is not ”tractable” by the HVBK two-fluid model. They have used a ”differential sub-grid closure” for the LES. Since, in the present research, we have shown that the mutual friction does not actually affect the way of energy cascade. And the source term brought by the mutual friction have a similar statistics behaviours to the dissipation rate. Maybe the Smagorinsky model might work with some simple tuning.

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