# The Gross-Pitaevskii equation on a two-dimensional ring: vortex nucleation and quantum turbulence

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# Context

## People

- Currently CEMPI postdoc (2022-2024).
  - PhD at Univ. Rennes (2019-2022), adv. Rémi Carles & Erwan Faou.
- Joint work with
  - mathematician Guillaume Dujardin (CR INRIA), Inria Lille.
  - physicists Radu Chicireanu (CR CNRS) and Jean-Claude Garreau (DR CNRS), PhLAM.

## Motivation

- Experimental setup at the PhLAM laboratory : cold atoms of potassium.
- Numerical simulations to highlight parameters for which we see **vortex formation**.
- Particular ring-shaped geometry in a two-dimensional setting.

# Cold atom trap at the PhLAM

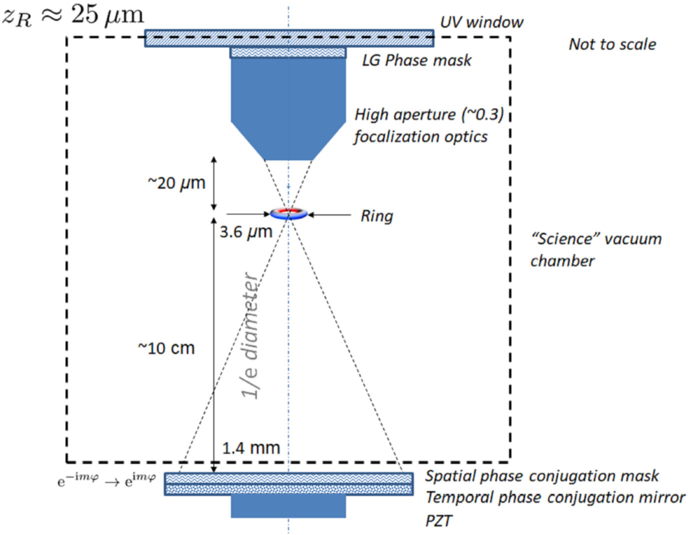



Figure – *Laguerre-Gauss beams as atom traps*, Jean-Claude Garreau.

# Gross-Pitaevskii equation

After proper sizing, simulation of the Gross-Pitaevskii equation

$$\begin{cases} i\partial_t \psi + \dfrac{1}{2m}\Delta\psi = \gamma|\psi|^2\psi + V\psi, \\ \psi(0,x) = \psi_0(x), \end{cases} \tag{GP}$$

for $t \geq 0$ and $x \in \Omega \subset \mathbb{R}^2$, where

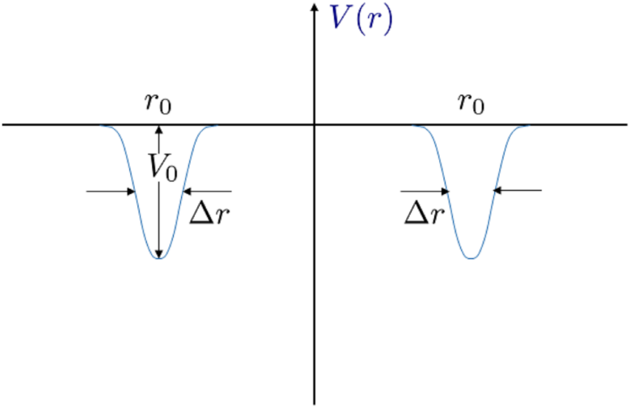
$$V(t,x) = V_{\rm pot}(x) + V_{\rm rot}(t,x).$$

- $V_{\rm pot}$ is a trapping potential, in which evolves the quantum fluid.
- $V_{\rm rot}$ is a rotating potential bringing energy to the system.

# Trapping potential

$$V_{\text{pot}}(r) = -V_0 \exp\left(-2m(r-1)^2\right)$$

with $V_0 > 0$ and $r = \sqrt{x^2 + y^2}$.



# Rotating potential

$$V_{\mathrm{rot}} = V_p V_{\mathrm{pot}}(r) \sin\left(n_\theta \theta - \Omega t\right),$$

with $0 \leq V_p \leq 1$, $n_\theta \in \mathbb{N}^*$ and angular velocity $\Omega$.
We will typically take $V_p \approx 0.05$ and $n_\theta = 6$.

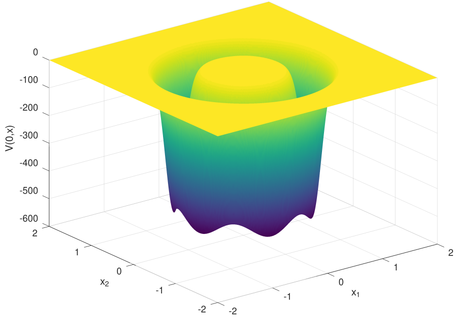



Figure – Potential $V_{\mathrm{rot}}$ at $t = 0$.

# Model

**Geometry**

$$V_{\mathrm{pot}}(r) \text{ is Gaussian} \Rightarrow |\psi|^2 \approx 0 \text{ for } r \gg 1 \text{ or } r \approx 0.$$

We then consider the ring

$$\Omega = \left\{ (x, y) \in \mathbb{R}^2 \;\middle|\; r_{\mathsf{min}} < \sqrt{x^2 + y^2} < r_{\mathsf{max}} \right\},$$

with $0 < r_{\mathsf{min}} < r_{\mathsf{max}} < 2$ and $r_{\mathsf{min}} + r_{\mathsf{max}} = 2$.

**Numerical methods**

- Finite Volume Scheme in space with Dirichlet conditions.
- Strang splitting method in time.
- Initial data : ground state solution for $V_{\mathrm{rot}} = 0$.

# Concentrated symmetric triangulation

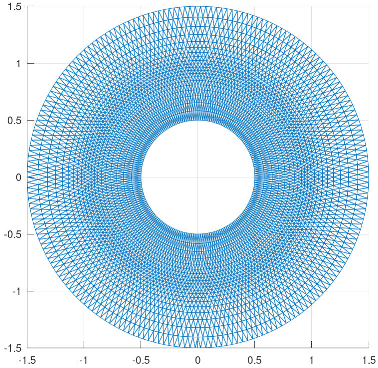



Figure – Concentrated symmetric triangulation with approximately 8000 triangles.

Still have to check that it satisfies the Delaunay condition (every $\theta < \pi/2$).

# Time integration

Splitting method

$$\partial_t \psi = A(\psi) + B(\psi) + C(\psi),$$

where

$$A(\psi) = \frac{i}{2m}\Delta\psi, \quad B(\psi) = -iV\psi, \quad C(\psi) = -i\gamma|\psi|^2\psi,$$

with associated flows

$$\begin{cases} u(t+\Delta t,\cdot) = \Phi_A^{\Delta t}(u(t,\cdot)) = e^{i\frac{\Delta t}{2m}\Delta} u(t,\cdot), \\ v(t+\Delta t,\cdot) = \Phi_B^{\Delta t}(v(t,\cdot)) = e^{-i\int_t^{t+\Delta t} V(s,\cdot)\mathrm{d}s} v(t,\cdot), \\ w(t+\Delta t,\cdot) = \Phi_C^{\Delta t}(w(t,\cdot)) = e^{-i\Delta t\gamma|w(t,\cdot)|^2} w(t,\cdot). \end{cases}$$

Strang splitting

$$\psi(t+\Delta t,\cdot) = \Phi_C^{\Delta t/2} \circ \Phi_B^{\Delta t/2} \circ \Phi_A^{\Delta t} \circ \Phi_B^{\Delta t/2} \circ \Phi_C^{\Delta t/2} \left(\psi(t,\cdot)\right).$$

## Computation of the ground state

We want to minimize the energy

$$E(\psi) = \int_\Omega \left( \frac{1}{2m} |\nabla\psi|^2 + V_{\mathrm{pot}}|\psi|^2 + \frac{\gamma}{2}|\psi|^4 \right) \mathrm{d}x$$

under the mass constraint

$$M(\psi) = \int_\Omega |\psi|^2 \mathrm{d}x = 1.$$

Writing $\psi(t,x) = e^{-i\mu t}\varphi(x)$ with $\mu \in \mathbb{R}$, we are brought back to the nonlinear eigenvalue problem

$$\begin{cases} \mu\varphi(x) = -\frac{1}{2m}\Delta\varphi(x) + V(x)\varphi(x) + \gamma|\varphi(x)|^2\varphi(x), & x \in \Omega, \\ \varphi(x) = 0, & x \in \partial\Omega, \\ \int_\Omega |\varphi(x)|^2 \mathrm{d}x = 1. \end{cases}$$

# Computation of the ground state

Semi-implicit **normalized gradient flow** of Faou Jézéquel 2018

$$\begin{cases} \dfrac{\varphi^{n+\frac{1}{2}} - \varphi^n}{\tau} = \dfrac{1}{2m} A\varphi^n - V\varphi^{n+\frac{1}{2}} - \gamma|\varphi^n|^2 \varphi^{n+\frac{1}{2}}, \\ \varphi^{n+1} = \dfrac{\varphi^{n+\frac{1}{2}}}{\|\varphi^{n+\frac{1}{2}}\|}, \end{cases}$$

with $\varphi^0 = \varphi_0(x)$ and step size $\tau > 0$. At each step, we check if

$$E\left(\varphi^{n+1}\right) < E\left(\varphi^n\right).$$

- If that is the case, move on until the stopping criterion ($\varepsilon > 0$ fixed)

$$\frac{\|\varphi^{n+1} - \varphi^n\|_{L^2(\Omega)}}{\tau} \leq \varepsilon.$$

- If not, we go back to $\varphi^n$ with $\tau \leftarrow \tau/2$.

# Dynamics

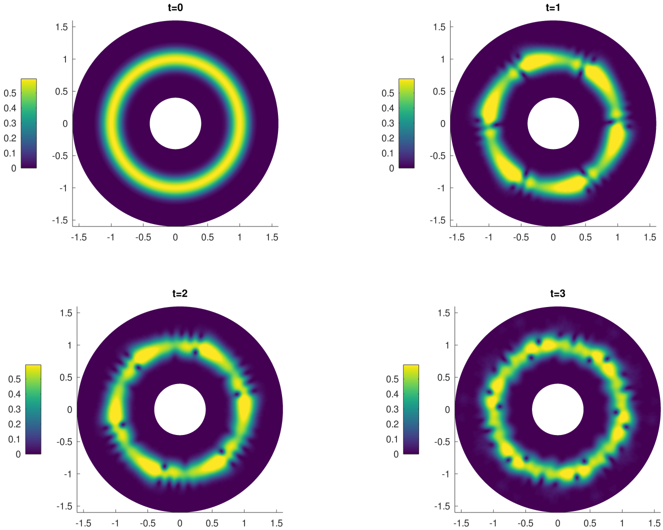



Figure – Density $|\psi(t,\cdot)|^2$ for several times $t$ ($m = 10$, $V_0 = 100$).

# Vortex detection algorithm

- Determine the potential centers of the vortices s.t. $|\psi(n)|^2 < tol_1$.
- We construct a list $\mathbb{P}$ :
  - For each potential vortex $n$, we look at the values of $|\psi|^2$ on adjacent triangles (at distance $\lambda = 1$).
  - If the values of $|\psi|^2$ are such that $|\psi|^2 - |\psi(n)|^2 > tol_2$, we add $n$ to $\mathbb{P}$, and we denote $\lambda_n = 1$ as the characteristic distance of $n$.
  - Otherwise, we repeat this procedure for triangles at distance $\lambda = 2, \dots, \lambda_{\max}$.

  At the end, for all $n \in \mathbb{P}$,

$$|\psi(n)|^2 < tol_1 \quad \text{and} \quad |\psi(j)|^2 > |\psi(n)|^2 + tol_2$$

  for each triangle $j \in \mathbb{S}_\lambda(n)$ at distance $1 \leq \lambda \leq \lambda_{\max}$.
- We consider each vortex $n$ of $\mathbb{P}$ and we look if another vortex belongs to the set $\bigcup_{\lambda=1}^{\lambda_{\max}} \mathbb{S}_\lambda(n)$. If so, we remove the vortex with the largest $|\psi|^2$ of the list $\mathbb{P}$.

# Vortex detection

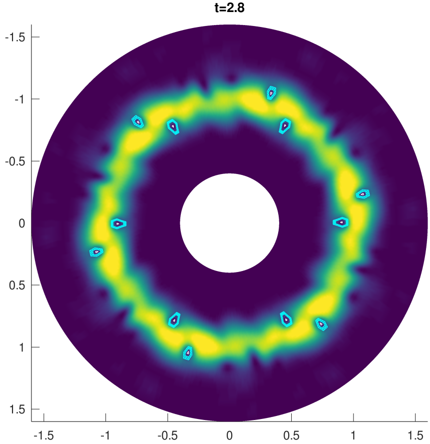
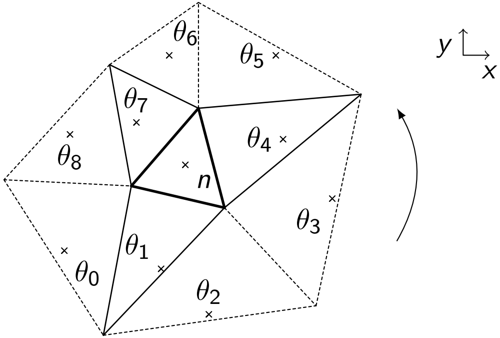



Figure – Detection algorithm applied at $t = 2.8$.

# Vortex indices

- For each $n \in \mathbb{P}$, we compute and sort the angles formed between circumcenters of triangle $n$ & of the triangles $j \in \mathbb{S}_\lambda(n)$ & the $x$ line.
- We compute $\theta_0 = \arg(\psi(j))$ associated to the triangle $j$ of lowest angle. We then proceed for other angles $\theta_1, \theta_2, \ldots, \theta_M$ :
  - After computing $\theta_m$, next angle $\tilde{\theta}_{m+1}$ is computed as an argument of the next value of $\psi$ on the set $\mathbb{S}_\lambda(n)$ with anticlockwise rotation.
  - We set $\theta_{m+1} := \tilde{\theta}_{m+1} + 2k\pi$ with $k = \mathrm{argmin}_{l\in\mathbb{Z}} |\tilde{\theta}_{m+1} - \theta_m + 2\pi l|$.
  - $\mathcal{I}(n) = \frac{\theta_M - \theta_0}{2\pi}$.



# Vortex detection

Table with position of the vortex, their characteristic distance and indices.

| $n$ | $\lambda(n)$ | $\mathcal{I}(n)$ |
|---|---|---|
| 14595 | 3 | -1 |
| 14757 | 3 | -1 |
| 14919 | 3 | -1 |
| 15081 | 3 | -1 |
| 15243 | 3 | -1 |
| 15405 | 3 | -1 |
| 26401 | 2 | 1 |
| 26563 | 2 | 1 |
| 26725 | 2 | 1 |
| 26887 | 2 | 1 |
| 27049 | 2 | 1 |
| 27211 | 2 | 1 |

# Conclusion

**Also**

- Decomposition of the wavefunction in the eigenbasis of the linearized operator.
- Computation of the velocity of the quantum fluid.

**Ongoing work**

- Understand the formation of vortices, compute their evolution.
- Full parametric study ($m$, $\gamma$, $V_0$, $V_p$, $n_\theta$, $\Omega$).
- Perform the experiment and compare with the numerics.

**Perspectives**

- Take into account dissipative phenomenon, loss of mass.

Thanks for your attention !