The Computation of Vortical Patterns in Bose-Einstein Condensates: Existence, Stability, Bifurcations and Dynamics

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Outline

• Motivation for Research
  – Wave Phenomena
  – Bose-Einstein Condensation

• From Newton’s Method to Deflation and Numerical Continuation
  – Deflated Continuation Method (DCM)

• Linear and Nonlinear Waves
  – The Nonlinear Schrödinger Equation (NLS)

• Bose-Einstein Condensates (BECs)
  – Mathematical Analysis of BECs using the NLS Equation
  – Solutions in 1D and 2D BECs
  – Discovery of Novel Solutions in 2D and 3D NLS Equations

• New Challenges and Future Research Directions
Wave Phenomena

Droplet

Ocean Waves

Gravitational Waves

Light Waves

Matter Waves
Bose-Einstein Condensates (BECs)

- State of matter in which a number of particles share the same quantum state.
- **1925**: Theoretical prediction by Bose & Einstein.
- **1995**: Experimental observation by Cornell, Ketterle, and Wieman.

- Everything condenses $\Rightarrow$ localized solution $\Rightarrow$ soliton!
From Newton’s Method to Deflation

- One of the most fundamental problems in Scientific Computing:
  \[ \text{Find } x^* \text{ such that } F(x) = 0. \]

- **Newton's method** constructs a sequence of iterates from an initial iterate \( x_0 \):
  \[ \{x_1, x_2, x_3, \ldots, x_n, \ldots \} \text{ such that } \lim_{n \to \infty} x_n = x^* \]
  via the iteration formula:
  \[ x_{n+1} = x_n - \frac{F(x_n)}{F'(x_n)} = x_n + p_n, \quad p_n := -\frac{F(x_n)}{F'(x_n)}, \quad n \geq 0. \]

- **Key advantage**: Generalizable to systems of equations \( \mathbf{F}(\mathbf{x}) = 0 \):
  \[ \mathbf{x}^{(n+1)} = \mathbf{x}^{(n)} - [\mathbf{J}(\mathbf{x}^{(n)})]^{-1} \mathbf{F}(\mathbf{x}^{(n)}), \quad n \geq 0. \]

- In practice, we solve a linear system:
  \[ -[\mathbf{J}(\mathbf{x}^{(n)})]^{-1} \mathbf{F}(\mathbf{x}^{(n)}) = \mathbf{p}^{(n)} \quad \Rightarrow \quad \begin{bmatrix} \mathbf{J}(\mathbf{x}^{(n)}) & \mathbf{p}^{(n)} \\ \mathbf{A} & \mathbf{x} & \mathbf{b} \end{bmatrix}. \]

- Plethora of methods and software for solving linear systems!

From Newton’s Method to Deflation

- **1963**: J.H. Wilkinson (1919-1986) proposed that if \( x_1, x_2, \ldots, x_k \) are roots of a polynomial \( p(x) \), further ones *may be found* by solving the **deflated problem**:

\[
q(x) = \frac{p(x)}{(x - x_1)(x - x_2) \cdots (x - x_k)}.
\]

- **1971**: K. Brown and W. Gearhart proposed that if \( x^* \) is a solution to \( F(x) = 0 \), a new solution *may be found* by solving:

\[
G(x) = M(x, x^*)F(x) = \frac{F(x)}{\|x - x^*\|},
\]

where \( M(x, y) = I/\|x - y\| \) is the **deflation matrix**.

- **2015**: P. Farrell, A. Birkisson and S.W. Funke introduced:

\[
G(x) = M(x, x^*)F(x), \quad M(x, y) := \left( \frac{1}{\|x - y\|^p + \sigma} \right) I.
\]

**Key properties:**
- For \( x \neq x^* \), \( G(x) = 0 \) iff \( F(x) = 0 \) (preservation of solutions of \( F \)).
- Newton’s method will not converge to \( x^* \) but to a **new solution**.
Numerical Continuation

- Let $F : U \times \mathbb{R} \mapsto V$ where $U$ and $V$ are Banach spaces.
- A common problem that arises in Scientific Computing is:
  \[
  F(x; \lambda) = 0.
  \]
- Use continuation methods to trace out branches of $x^*$ as $\lambda$ is varied.
- Commonly used methods:

  **Natural/Sequential**

  ![Diagram of Natural/Sequential Continuation](image)

  **Pseudo-arclength**

  ![Diagram of Pseudo-arclength Continuation](image)

[Krauskopf, Osinga & Galán-Vioque, *Numerical Continuation Methods for Dynamical Systems* (Springer-Verlag, 2007)]

[Y. Kuznetsov, *Elements of Applied Bifurcation Theory* (Springer-Verlag, 2023)]
Numerical Continuation: Using Pseudo-Arclength

Consider the root-finding problem:

\[ F(x; \lambda) = x^4 - 5x^2 - \lambda x + 5 + 0.5\lambda. \]
Numerical Continuation: Using Pseudo-Arclength

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- Pseudo-arclength continuation fails here.

Another Branch!
Deflated Continuation Method (DCM)

- DCM enables the discovery of previously unknown disconnected branches of solutions.
- Given $F(u; \lambda)$, employ Newton's method with fixed $\lambda$.
- Upon convergence $\Rightarrow u^*$ is obtained.
- Now, find a new solution $\Rightarrow$ deflate $u^*$.
- Construct and solve a new nonlinear problem:

$$G(u) = 0.$$ 

with

$$G(u) \doteq M(u; u^*)F(u), \quad M(u; u^*_1) \doteq (\|u - u^*_1\|^{-2} + 1) I.$$
A DCM Example

- Consider the root-finding problem:

\[ F(x; \lambda) = x^4 - 5x^2 - \lambda x + 5 + 0.5\lambda. \]

- DCM with initial guess \( x^{(0)} = -0.1 \) and \( \lambda = -8 \):
A DCM Example

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![Graph showing the root-finding process for DCM example](image)

Trace the entire branch!
Mathematical Modeling of Wave Phenomena

Droplet

Ocean Waves

Gravitational Waves

Light Waves

Matter Waves

Courtesy: Pink Floyd
Linear Waves

- Matter wave is described by \( \Phi(r, t) \in \mathbb{C} \) with \( r = \langle x, y, z \rangle \).
- The probability density \( |\Phi|^2 \) is normalized according to

\[
\int_{\mathbb{R}^3} |\Phi|^2 \, d\mathbf{r} = 1.
\]

- **1926:** Matter Waves can be described by Erwin Schrödinger’s Equation:

\[
i \frac{\partial \Phi}{\partial t} = \hat{H}_0 \Phi, \quad \hat{H}_0 = -\frac{1}{2} \nabla^2 + V(r), \quad i = \sqrt{-1},
\]

\[
\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.
\]

- \( V(r) \) is the external potential.
- Solve this **linear** partial differential equation (PDE) using separation of variables:

\[
\Phi(r, t) = \phi(r) e^{-iEt}.
\]

- Obtain an **eigenvalue problem**:

\[
\hat{H}_0 \phi = E \phi.
\]
Example: Quantum Harmonic Oscillator (QHO)

• In 1D, the external potential is:

\[ V(x) = \frac{1}{2} \Omega^2 x^2, \]

• The eigenvalue problem becomes:

\[
-\frac{1}{2} \frac{d^2 \phi(x)}{dx^2} + \frac{1}{2} \Omega^2 x^2 \phi(x) = E \phi(x) \quad (u = \sqrt{\Omega} x)
\]

\[
\frac{d^2 \phi}{du^2} + \left( \frac{2E}{\Omega} - u^2 \right) \phi = 0.
\]

• For \(|u| \gg 1\), \(\phi(u) \to 0\), and thus we get:

\[
\frac{d^2 \phi}{du^2} - u^2 \phi = 0, \quad \text{with solution} \quad \phi(u) \propto u^k e^{-u^2/2}.
\]

• Setting \(\phi(u) = f(u)e^{-u^2/2}\) we get the Hermite differential equation:

\[
\frac{d^2 f}{du^2} - 2u \frac{df}{du} + 2nf = 0, \quad \text{with} \quad \frac{2E}{\Omega} - 1 = 2n \Rightarrow E_n = (n + 1/2) \Omega.
\]
Example: Quantum Harmonic Oscillator (QHO)

- Solutions to Hermite equation are the Hermite polynomials $H_n$:

\[
\begin{align*}
H_0(u) &= 1 \\
H_1(u) &= 2u \\
H_2(u) &= 4u^2 - 2
\end{align*}
\]

\[
\Rightarrow \text{obtained by using power series: } f(u) = \sum_{j=0}^{\infty} \alpha_j u^j.
\]

- The solutions to the QHO in 1D are given by:

\[
\phi_n(x) = \left( \frac{\Omega}{\pi} \right)^{1/4} \sqrt{\frac{1}{2^n n!}} H_n \left( \sqrt{\Omega} x \right) e^{-\Omega x^2 / 2}.
\]
The QHO in 2D: Cartesian Eigenstates

- The Linear Schrödinger equation in 2D takes the form:

\[-\frac{1}{2} \nabla^2 \phi(r) + \frac{1}{2} \Omega^2 \left(x^2 + y^2\right) \phi(r) = E \phi(r), \quad r = \langle x, y \rangle.\]

- Solution of the Sturm-Liouville problem in Cartesian coordinates:

\[|m, n \rangle = \phi_{m,n}(r) \propto H_m \left(\sqrt{\Omega} x\right) H_n \left(\sqrt{\Omega} y\right) e^{-\Omega r^2/2}, \quad E_{m,n} = (m + n + 1) \Omega.\]

- Probing the density \(|\phi_{m,n}|^2|\):
The QHO in 2D: Polar Eigenstates

1. The Linear Schrödinger equation in 2D but with $\phi(x, y) = q(r)e^{il\theta}$ reads:

$$-\frac{1}{2} \left( \frac{d^2 q}{dr^2} + \frac{1}{r} \frac{dq}{dr} - \frac{l^2 q}{r^2} \right) + \frac{1}{2} \Omega^2 r^2 q = E q.$$ 

2. Solution of the Sturm-Liouville problem in Polar coordinates:

$$|k, l\rangle = \phi_{k,l}(r, \theta) \propto q_{k,l} e^{il\theta}, \quad E_{k,l} = (1 + |l| + 2k) \Omega.$$ 

with $q_{k,l} \propto r^l L_k^l (\Omega r^2) e^{-\Omega r^2 / 2}$ ($L_k^l$ are the Laguerre polynomials).

3. Probing the density $|\phi_{k,l}|^2$:

![Graphs of $|\phi_{k,l}|^2$ for different values of $k$ and $l$.]
The QHO in 3D

- Cartesian eigenfunctions:

\[ |k, m, n\rangle \propto H_k(\sqrt{\Omega}x) H_m(\sqrt{\Omega}y) H_n(\sqrt{\Omega}z)e^{-\Omega r^2/2}, \quad E_{m,n} = (k + m + n + 3/2) \Omega. \]

- Cylindrical eigenfunctions:

\[ |K, l, n\rangle \propto q_{K,l}(R)e^{il\theta} H_n \left(\sqrt{\Omega}z\right)e^{-\Omega(R^2+z^2)/2}, \quad E_{K,l,n} = (2K + |l| + n + 3/2) \Omega. \]

with \( R = \sqrt{x^2 + y^2} \) and \( q_{K,l} = R^l L^l_K(\Omega R^2) e^{-\Omega R^2/2}. \)

- Spherical eigenfunctions:

\[ |K, l, m\rangle \propto q_{K,l}(r)Y_{l,m}(\theta, \phi), \quad E_{K,l,m} = (2K + l + 3/2) \Omega, \]

with \( Y_{l,m} \) the spherical harmonics, \( r = \sqrt{x^2 + y^2 + z^2} \), and \( m = 0, \pm 1, \ldots, \pm l. \)
The QHO in 3D

- Examples of eigenfunctions in 3D:
  - **Vortex Line (VL):** $u_{VL} \propto |0, 2, 0\rangle$ in cylindrical coordinates.
  - One dark soliton ($z = 0$): $u_{DS} \propto |0, 0, 1\rangle$.
  - Ring dark soliton (RDS): $u_{RDS} \propto |2, 0, 0\rangle + |0, 2, 0\rangle$.
  - **Vortex Ring (VR):** $u_{VR} = u_{RDS} + iu_{DS}$.
Bose–Einstein Condensates (BECs)

- State of matter in which a number of particles share the same quantum state.
- **1925**: Theoretical prediction by Bose & Einstein.
- **1995**: Experimental observation by Cornell, Ketterle, and Wieman.

- Everything condenses ⇒ localized solution ⇒ soliton!
The Nonlinear Schrödinger (NLS) Equation and BECs

- The NLS can be used to describe light propagation in nonlinear optics, water waves and Bose-Einstein Condensates (BECs):

\[
i \frac{\partial \Phi(r, t)}{\partial t} = \left[ -\frac{1}{2} \nabla^2 + V(r) + \gamma |\Phi(r, t)|^2 \right] \Phi(r, t).
\]

- External potential:

\[
V(r) = \frac{1}{2} \Omega^2 |r|^2.
\]

- \( \gamma = -1 \): Attractive interactions.
- \( \gamma = 1 \): Repulsive interactions.
- \( |\Phi(r, t)|^2 \) describes atomic density in a condensate.
- **Nonlinearity** due to the interatomic interaction.

The NLS with $V \equiv 0$: Exact Solutions

- This is a special type of a PDE:

$$i \frac{\partial \Phi}{\partial t} = -\frac{1}{2} \frac{\partial^2 \Phi}{\partial x^2} + \gamma |\Phi|^2 \Phi, \quad \gamma = \pm 1.$$

- Bright soliton for $\gamma = -1$:

$$\Phi(x, t) = A \text{sech} \left[ A (x - x_0) \right] e^{i(A^2/2)t}, \quad \mu = -A^2/2.$$
The NLS with \( V \equiv 0 \): Exact Solutions

- Dark soliton for \( \gamma = 1 \):
  \[
  \Phi(x, t) = \sqrt{\mu} \tanh \left( \sqrt{\mu} (x - x_0) \right) e^{-i\mu t}.
  \]

- There exist conserved quantities:
  
  \[
  N = \int_{\mathbb{R}} |\Phi|^2 dx \quad \text{(Number of atoms)}
  \]
  
  \[
  P = \frac{i}{2} \int_{\mathbb{R}} (\Phi \Phi^* - \Phi^* \Phi_x) \, dx \quad \text{(Momentum)}
  \]
  
  \[
  E = \frac{1}{2} \int_{\mathbb{R}} \left( |\Phi_x|^2 + \gamma |\Phi|^4 \right) \, dx \quad \text{(Energy)}
  \]
Mathematical Analysis of BECs using the NLS Equation

• Constructing solutions to the NLS using the ansatz:

\[ \Phi(r, t) = \phi(r)e^{-i\mu t}. \]

• Steady-state problem:

\[ -\frac{1}{2}\nabla^2 \phi + |\phi|^2 \phi + V(r)\phi - \mu \phi = 0. \]

• Special cases:
  
  • The non-interacting case \( \Rightarrow |\phi|^2 \approx 0 \Rightarrow \text{Quantum Harmonic Oscillator}: \)

  \[ -\frac{1}{2}\nabla^2 \phi + V(r)\phi = \mu \phi. \]

  • Slow spatial variations of \( |\phi|^2 \) results in \( \nabla^2 \phi \approx 0 \Rightarrow \text{Thomas-Fermi limit}: \)

  \[ |\phi(r)|^2 = \begin{cases} 
  \mu - V(r), & \mu > V(r), \\
  0, & \text{otherwise}. 
\end{cases} \]

• Fundamental question: What is happening between those two limits?
Mathematical Analysis of BECs using the 1D NLS

• Numerical solutions of the 1D NLS:

We monitor: \(N = \int_D |\phi(x)|^2 \, dx\).
Mathematical Analysis of BECs using the 2D NLS

- Numerical solutions of the 2D NLS:

- We monitor: $N = \int_{D} |\phi(x, y)|^2 dx dy$. 
Deflated Continuation Method & Bifurcation Analysis

- Consider the time-dependent NLS:

\[
i \frac{\partial \Phi(r, t)}{\partial t} = \left[ -\frac{1}{2} \nabla^2 + V(r) + |\Phi(r, t)|^2 \right] \Phi(r, t),
\]

and the perturbation ansatz around \( \phi_0(r) \):

\[
\Phi(r, t) = e^{-i\mu t} \{ \phi_0(r) + \varepsilon \left[ a(r)e^{i\omega t} + b^*(r)e^{-i\omega^* t} \right] \}, \quad \varepsilon \ll 1.
\]

- At order \( \mathcal{O}(\varepsilon) \), we obtain the eigenvalue problem:

\[
\omega \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -\mathcal{L} & \phi_0^2 \\ -\phi_0^2 & -\mathcal{L} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}, \quad \mathcal{L} = -\frac{1}{2} \nabla^2 + 2|\phi_0|^2 + V(r) - \mu.
\]

\[\text{Stable} \quad \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\text{Exponentially Unstable}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\text{Oscillatorily Unstable}
\end{array}
\end{array}
\end{array}\]

\[\text{[E.G. Charalampidis, P.G. Kevrekidis, P. Farrell, CNSNS (2018)]}\]
Deflated Continuation Method for the 2D NLS

- Bifurcation Analysis: Benchmarking of the DCM.

[E.G. Charalampidis, P.G. Kevrekidis, P. Farrell, CNSNS (2018)]
Deflated Continuation Method for the 2D NLS

- Bifurcation Analysis: Benchmarking of the DCM [Video].

[E.G. Charalampidis, P.G. Kevrekidis, P. Farrell, CNSNS (2018)]
Deflated Continuation Method for the 2D NLS

- DCM Solutions: 63 solutions found, including 15 new ones.

[E.G. Charalampidis, P.G. Kevrekidis, P. Farrell, CNSNS (2018)]
DCM for the 2D NLS: Discovery of New Solutions

- Few solutions that had **not** been identified before.

- System prefers to create Vortical Patterns [Video].

  [E.G. Charalampidis, P.G. Kevrekidis, P. Farrell, CNSNS (2018)]
DCM for Multicomponent NLS: The 2D case

• A two-component NLS system in 2D:

\[ i \frac{\partial \Phi_-}{\partial t} = -\frac{D_-}{2} \nabla^2 \Phi_- + \left( g_{11} |\Phi_-|^2 + g_{12} |\Phi_+|^2 \right) \Phi_- + V(r) \Phi_- , \]
\[ i \frac{\partial \Phi_+}{\partial t} = -\frac{D_+}{2} \nabla^2 \Phi_+ + \left( g_{12} |\Phi_-|^2 + g_{22} |\Phi_+|^2 \right) \Phi_+ + V(r) \Phi_+ . \]

• Seeking for steady-state solutions:

\[ \Phi_\pm (r, t) = \phi_\pm (r) e^{-i\mu \pm t} . \]

• Obtain a steady-state problem:

\[ -\frac{D_-}{2} \nabla^2 \phi_- + \left( g_{11} |\phi_-|^2 + g_{12} |\phi_+|^2 \right) \phi_- + V(r) \phi_- - \mu_- \phi_- = 0 , \]
\[ -\frac{D_+}{2} \nabla^2 \phi_+ + \left( g_{12} |\phi_-|^2 + g_{22} |\phi_+|^2 \right) \phi_+ + V(r) \phi_+ - \mu_+ \phi_+ = 0 . \]

• Fix \( D_- = D_+ = 1 \), \( g_{11} = 1.03 \), \( g_{22} = 0.97 \), \( g_{12} = 1 \), \( \mu_- = 1 \), \( V(r) = \Omega^2 |r|^2 / 2 \) with \( \Omega = 0.2 \).

• Continuation parameter: \( \mu_+ \).

[EGC, N. Boullé, P.G. Kevrekidis, P. Farrell, CNSNS (2020)]
DCM for Multicomponent NLS: New Results
DCM for Multicomponent NLS: New Results

[EGC, N. Boullé, P.G. Kevrekidis, P. Farrell, CNSNS (2020)]
DCM for Multicomponent NLS: New Results
DCM for Multicomponent NLS: New Results

(c) 

(d)
DCM for Multicomponent NLS: New Results

[EGC, N. Boullé, P.G. Kevrekidis, P. Farrell, CNSNS (2020)]
State-Of-The-Art Eigenvalue Solver: FEAST

- Stability matrix $A$ is a $357,604 \times 357,604$ sparse matrix containing 2,856,048 non-zero elements.
- Initially, the spectra were computed by using MATLAB's `eigs` built-in command.
- Spurious instabilities appear in the spectrum:

![Graph showing spurious instabilities](image)

- This observation was validated by computing:

\[
\left\| A W_R - \rho W_R \right\|_1 / \| A \|_1.
\]

- The above formula for $\mu_+ = 1.3105$ gives $\approx 44.72$.

[EGC, N. Boullé, P.G. Kevrekidis, P. Farrell, CNSNS (2020)]
State-Of-The-Art Eigenvalue Solver: FEAST

- Next, we used the Multiprecision Computing Toolbox “Advanpix” with 34 digits.
- The $l_2$-norm for 100 eigenpairs $(\rho, W_R)$ was $\approx 7.3 \times 10^{-18}$.
- The computation of the spectra of a single branch (121 distinct values in $\mu_+$) took $\sim 3$ months.
- FEAST combines accuracy, efficiency and robustness while exhibiting natural parallelism at multiple levels.
- Comparison between FEAST and Multiprecision Computing Toolbox:

[EGC, N. Boullé, P.G. Kevrekidis, P. Farrell, CNSNS (2020)]
DCM for the single-component 3D NLS: Exotic Yet New Solutions

[Video (VR + VL “handles”)]

[N. Boullé, EGC, P. Farrell, P.G. Kevrekidis, PRA (2020)]
DCM for the single-component 3D NLS: Exotic Yet New Solutions

[Video (5VLs + 2VRs)] [Video (S-VR type)]

[N. Boullé, EGC, P. Farrell, P.G. Kevrekidis, PRA (2020)]
DCM for the single-component 3D NLS: Exotic Yet New Solutions

[N. Boullé, EGC, P. Farrell, P.G. Kevrekidis, PRA (2020)]
Multicomponent NLS systems: Using PS in 3D

- Spinor 3D BECs:

\[
\begin{align*}
 i \frac{\partial \psi_{+1}}{\partial t} &= \mathcal{H}_{+1} + c_2 (|\psi_0|^2 + F_z) \psi_{+1} + c_2 \psi_{-1}^* \psi_0^2, \\
 i \frac{\partial \psi_0}{\partial t} &= \mathcal{H}_0 + c_2 (|\psi_{+1}|^2 + |\psi_{-1}|^2) \psi_0 + 2c_2 \psi_0^* \psi_{+1} \psi_{-1}, \\
 i \frac{\partial \psi_{-1}}{\partial t} &= \mathcal{H}_{-1} + c_2 (|\psi_0|^2 - F_z) \psi_{-1} + c_2 \psi_{+1}^* \psi_0^2, \\
 \mathcal{H} &= -\frac{1}{2} \nabla^2 + V(\mathbf{r}) + c_0 \sum_{m=-1}^{1} |\psi_m|^2.
\end{align*}
\]

- A saddle-center bifurcation was found through pseudo-arclength continuation:

\[\text{[M. Thudiyangal, R. Carretero-González, EGC, D.S. Hall, P.G. Kevrekidis, PRA (2022)]}\]
Multicomponent 3D NLS systems: New Solutions

- Two Alice-Ring solutions were found using pseudo-arclength continuation:

[M. Thudiyangal, R. Carretero-González, EGC, D.S. Hall, P.G. Kevrekidis, PRA (2022)]
New Challenges and Future Research Directions

- Existing tools for bifurcation analysis of complex nonlinear systems may fail to detect disconnected branches of solutions.
- The DCM can become a robust computational tool for discovering new solutions and studying their bifurcations and stability.

**Proposed Project**: Bifurcation Tools in FreeFEM++ for Robust Bifurcation and Stability Analysis of Complex Nonlinear Systems.

- Implementation of DCM in FreeFEM++ with domain-decomposition techniques in parallel computing platforms.
- Implementation of pseudo-arclength & Induced Dimension Reduction method (IDR) in FreeFEM++. It outperforms BI-CGSTAB!
- Integrate the state-of-the-art FEAST eigenvalue solver for solving extremely large yet ill-conditioned eigenvalue problems.
- Ongoing collaboration with the Numerical Analysis group in Rouen of Prof. Ionut Danaila & Dr. Georges Sadaka, and with the experimental group of Prof. David Hall (Physics & Astronomy, Amherst College).
New Challenges and Future Research Directions

• Experimental results on a five-component 3D NLS system:

Credit: Prof. David Hall (Physics & Astronomy, Amherst College)
Collaborators

• Panayotis Kevrekidis, UMass Amherst
• David Hall, Amherst College
• Patrick Farrell, Oxford University
• Nicolas Boullé, Cambridge University
• Ricardo Carretero-González, San Diego State University
• Thudiyangal Mithun, University of Luxembourg
• Avadh Saxena, Los Alamos National Laboratory
• Fred Cooper, Santa Fe Institute & Los Alamos National Laboratory
• Ionut Danaila & Georges Sadaka, Université de Rouen Normandie
• Pierre Jolivet, Sorbonne Université
• Boris Malomed, Tel Aviv University

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• US-Israel Binational Science Foundation, Grant No. 2010239
Key References


- F. Cooper, A. Khare, E.G. Charalampidis, A. Saxena, PRE 107 (2023) 064202.
The QHO in 2D: Cartesian Coordinates

- The Linear Schrödinger equation in 2D takes the form:

\[-\frac{1}{2} \nabla^2 \phi(r) + \frac{1}{2} \Omega^2 (x^2 + y^2) \phi(r) = E \phi(r) , \quad r = \langle x, y \rangle .\]

- Upon writing \( \phi(r) = X(x)Y(y) \), we obtain two 1D QHO equations:

- \( -\frac{1}{2} \frac{d^2 X}{dx^2} + \frac{1}{2} \Omega^2 x^2 X = E_m X , \)

- \( -\frac{1}{2} \frac{d^2 Y}{dy^2} + \frac{1}{2} \Omega^2 y^2 Y = E_n Y , \)

where \( E_m = (m + 1/2) \Omega \) and \( E_n = (n + 1/2) \Omega \) \( \Rightarrow E_{m,n} = (m + n + 1) \Omega . \)

- Density \( |\phi_{m,n}|^2 \) of solutions:
Key References