Vortices and vortex rings in quantum superfluids: a quasi-particle approach

Ricardo Carretero

Nonlinear Dynamical Systems Group
http://nlds.sdsu.edu/
Computational Science Research Center
Department of Mathematics and Statistics
San Diego State University

Collaborators/Links (I)

- Nonlinear Dynamical Systems @ SDSU: http://nlds.sdsu.edu/
  - Peter Blomgren (Numerical PDEs, image processing)
  - Ricardo Carretero (App. math., nonlinear lattices and waves)
  - Chris Curtis (Nonlinear Waves, fluids, optics)
  - Jérôme Gilles (Imaging, wavelets, signal/image processing)
  - Joe Mahaffy (Mathematical biology, delay differential equations)
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  - New hire 2023 (Mathematical Data Science, dynamical systems)

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- Research Students involved in nonlinear waves
  - Miko Stulajter (PhD, Comp. Sci.).
  - Seth Minor, E. Bartley, J. Borjon, (MS, Dyn. Syst.).
  - Recent graduates working on coherent structures: Z. Maches (19), B. Hoang (19), M. Richards (17), J. Rossi (17), J. Talley (17), C. Prieto (16), M. Kandes (16), V. Berardi (15), R. Navarro (13), W. Henry (13), K. Joiner (13), R. Caplan (12), E. Baik (12).

Collaborators in Nonlinear Waves/Lattices (II)

- David Hall (Amherst Coll.)
- Daniele Sanvitto (Lecce)
- Lorenzo Dominici (Lecce)
- Peter Engels (WSU)
- Brian Anderson (UoA)
- Chiara Daraio (Caltech)
- Lars English (Dickinson)
- Panos Kevrekidis (UMass)
- D. Frantzeskakis (Athens)
- Ionut Danaila (Rouen)
- Stathis Charalampidis (CalPoly)
- Nick Proukakis (Newcastle)
- Tasso Kaper (BU)
- Nate Whitaker (UMass)
- Chris Curtis (SDSU)
- Luis Cisneros-Ake (IPN)
- Wenlong Wang (Chengdu)
- Theo Kolokolnikov (Dalhousie)
- Boris Malomed (Tel Aviv)
- Roy Goodman (NJIT)
- Nathan Kutz (UoW)
- Bernard Deconinck (UoW)
- Dan Spirn (Minnesota)
- Mason Porter (UCLA)
- Yuri Kivshar (Camberra)
- W. Królikowski (TAMU-Qatar)
- Jesús Cuevas (Sevilla)
- Peter Schmelcher (Heidelberg)
- Klaus Sengstock (Hamburg)
- Vladimir Konotop (Lisboa)
- Pedro Torres (Granada)
- Todd Kapitula (Calvin Col.)
- Keith Promislow (MSU)
- Bernard Deconinck (UoW)
- Lincoln Carr (Col. Sch. Mines)
- Ashton Bradley (Otago)
- Jared Bronski (UI-UC)
- etc, ... (>150)

Road map

- Introduction
  - BECs, External trapping $\rightarrow$ controlling dimensionality
  - Motivation: physical experiments

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  - Vortex pairs $(+,+)$ $\rightarrow$ Bifurcation analysis
  - Bifurcation in the actual experiment!!!
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  - Anisotropy $\rightarrow$ complex periodic orbits
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- Going 3D $\rightarrow$ Vortex rings (VRs)
  - VR $\rightarrow$ self-induced velocity
  - VR-VR interactions $\rightarrow$ Biot-Savart law
  - Leapfrogging dynamics of VRs
  - VR scattering (co-planar)

Gross-Pitaevskii Eq.:

Close to $T = 0$ BEC $\rightarrow$ Gross-Pitaevskii Eq.:

$$i\hbar\frac{\partial\psi}{\partial t} = \left[-\frac{\hbar^2}{2m}\nabla^2 + V_{\text{ext}}(\mathbf{r}) + g\left|\psi\right|^2\right]\psi,$$

- $\psi(x, y, z, t)$: wavefunction,
  $|\psi|^2$: density,
  $N = \iiint |\psi|^2\, dV$: # of atoms,
- nonlinear coeff: $g = 4\pi\hbar^2 a_s/m$, where a_s scattering length:
  - a_s>0 : repulsive : (^{23}Na, ^{87}Rb, H, ^{4}He, ^{85}Rb)$\rightarrow$ [DSs, vortices, VRs, VLs]
  - a_s<0 : attractive : (^{7}Li, ^{85}Rb)$\rightarrow$ [BSs, Bose Nova]

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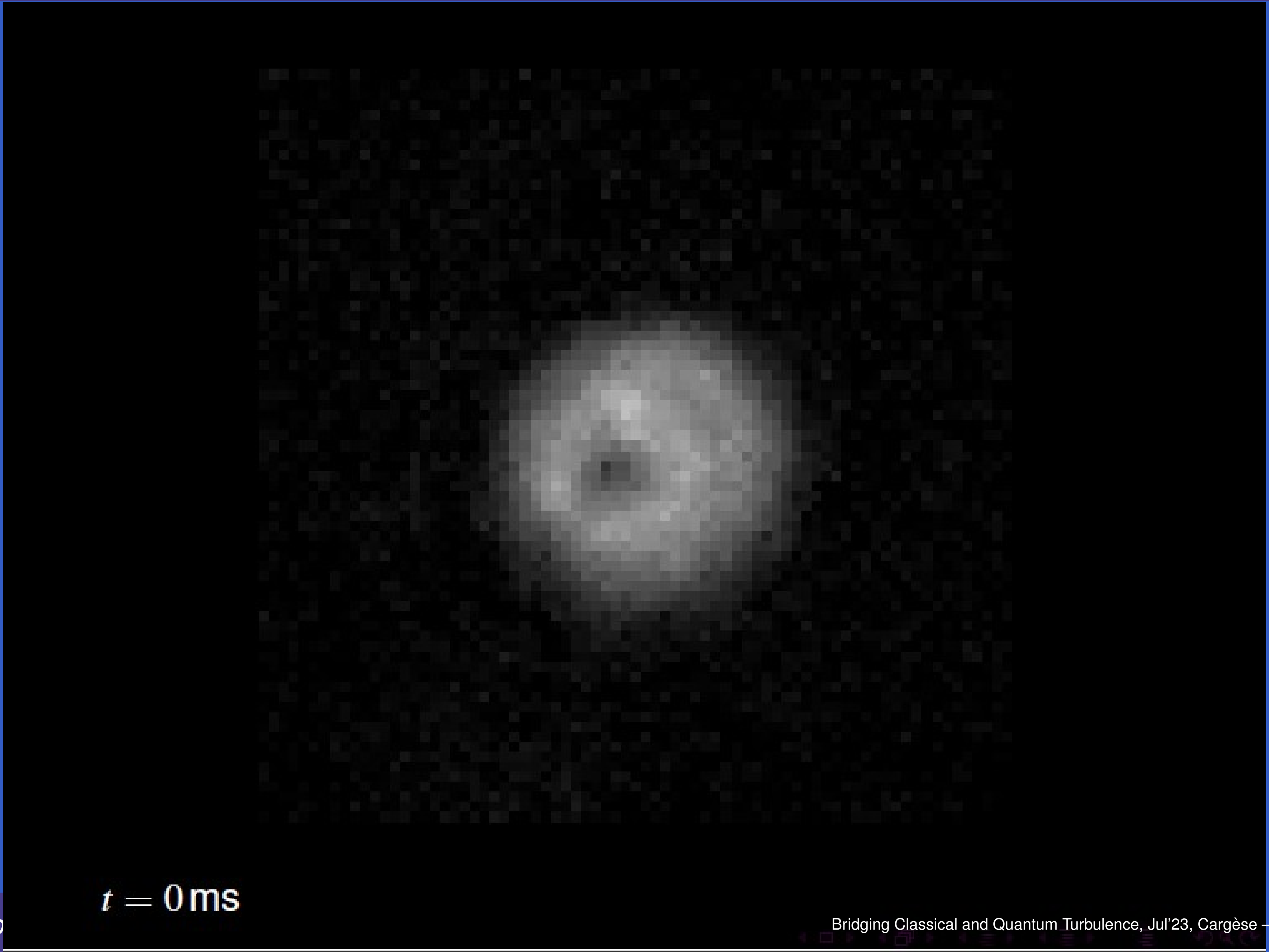
- External potential: $V_{\mathrm{ext}}(\mathbf{r}) = \frac{1}{2}m\omega_r^2 r^2 + \frac{1}{2}m\omega_z^2 z^2$
  - $\omega_r \gg \omega_z \rightarrow$ 1D BEC $\rightarrow$ bright/dark solitons
  - $\omega_z \gg \omega_r \rightarrow$ 2D BEC $\rightarrow$ vortices
  - $\omega_r \approx \omega_z \rightarrow$ 3D BEC $\rightarrow$ vortex lines/vortex rings

Vortex dynamics in EXPERIMENTAL BECs (David Hall's experiment)

Vortex precession

Experimental BEC vortices: precession in MT



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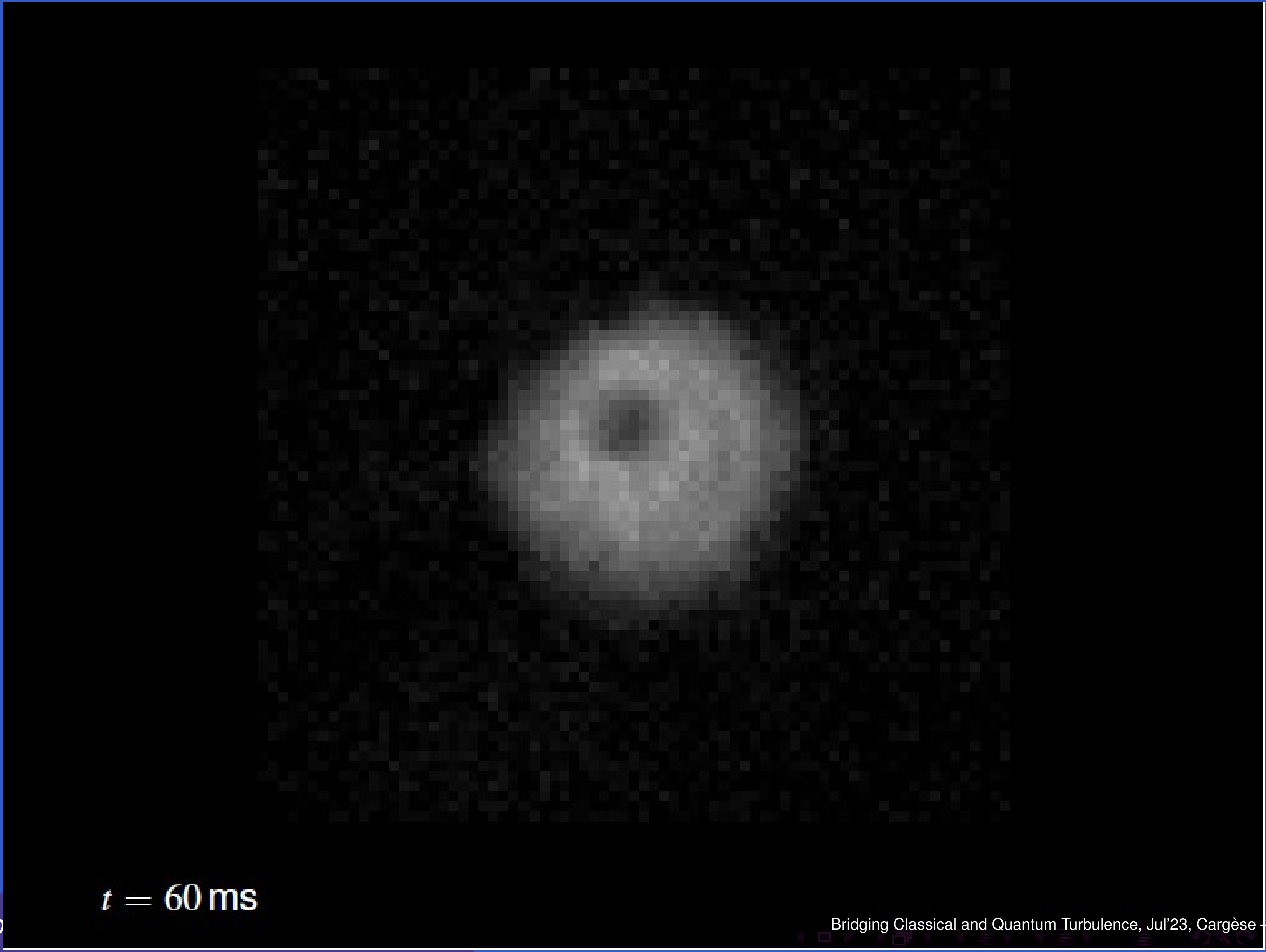

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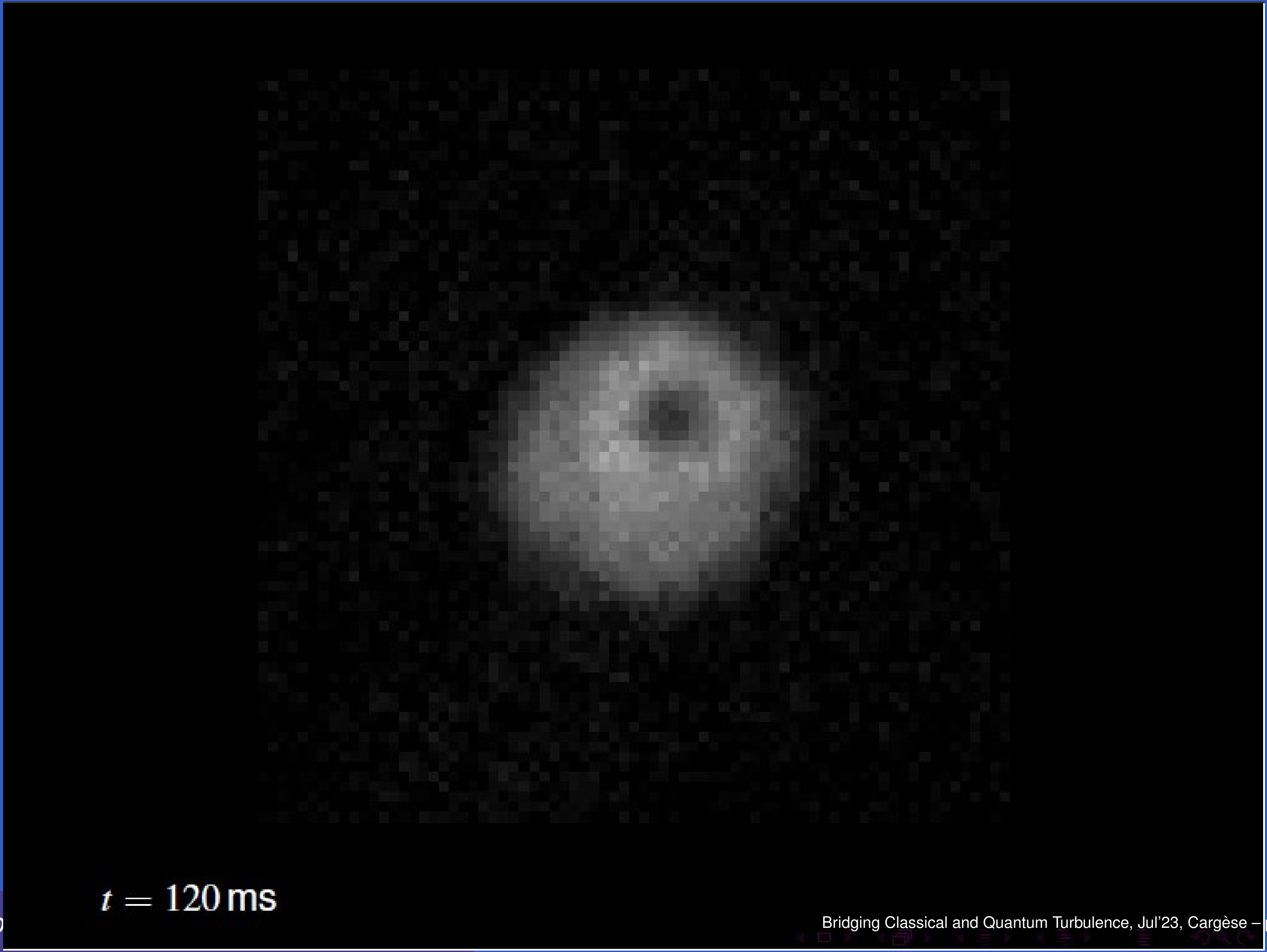

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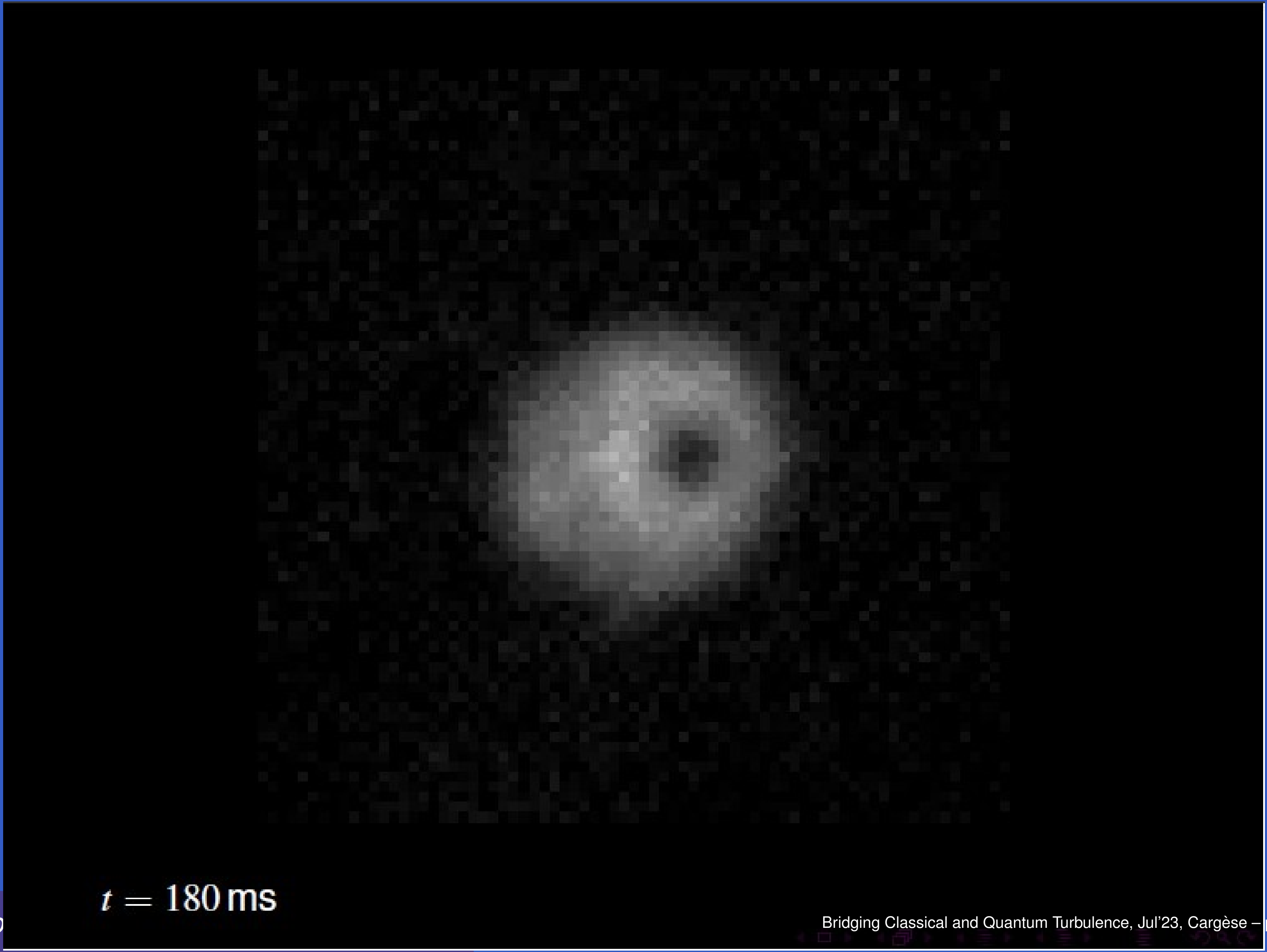

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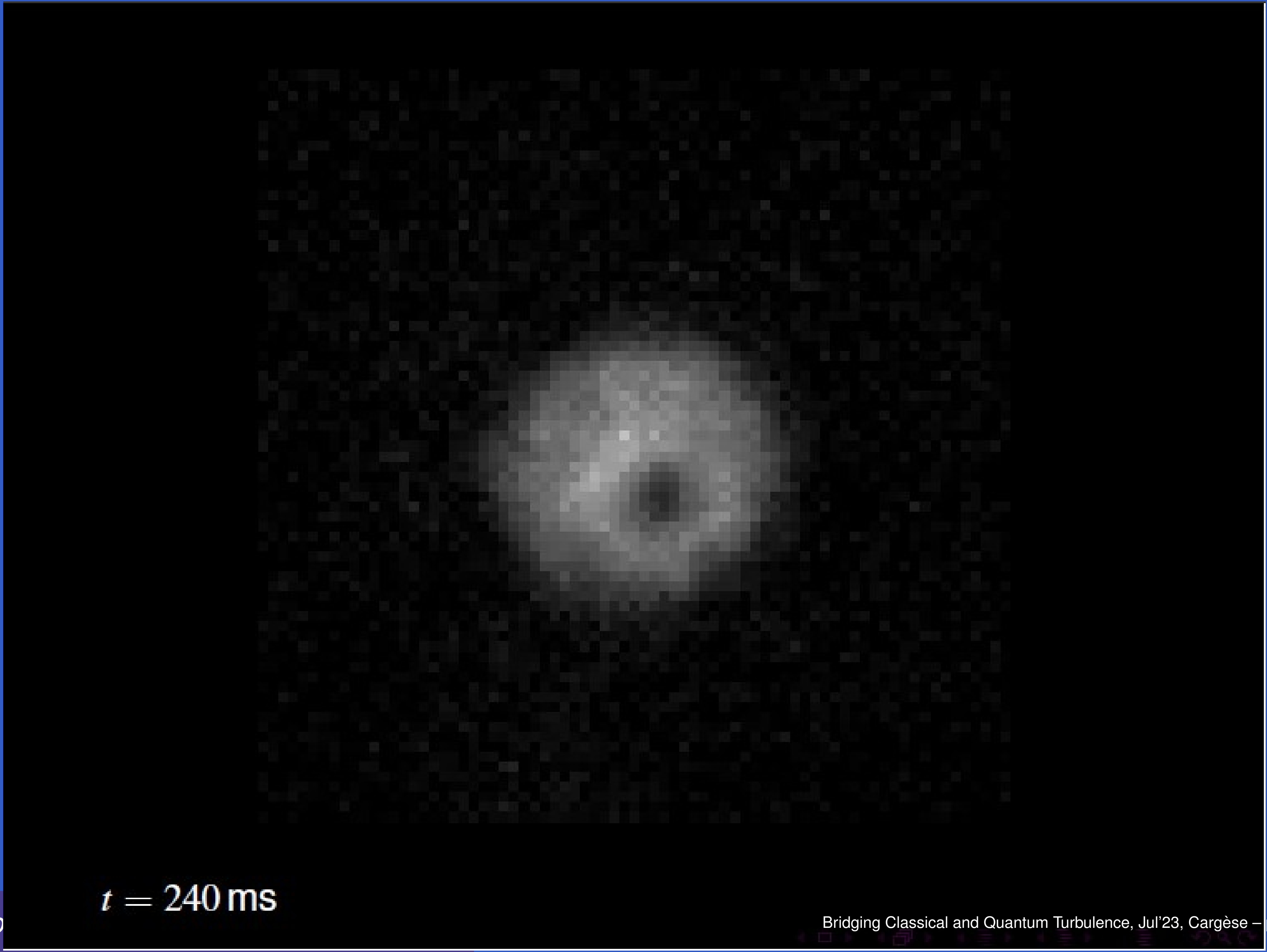

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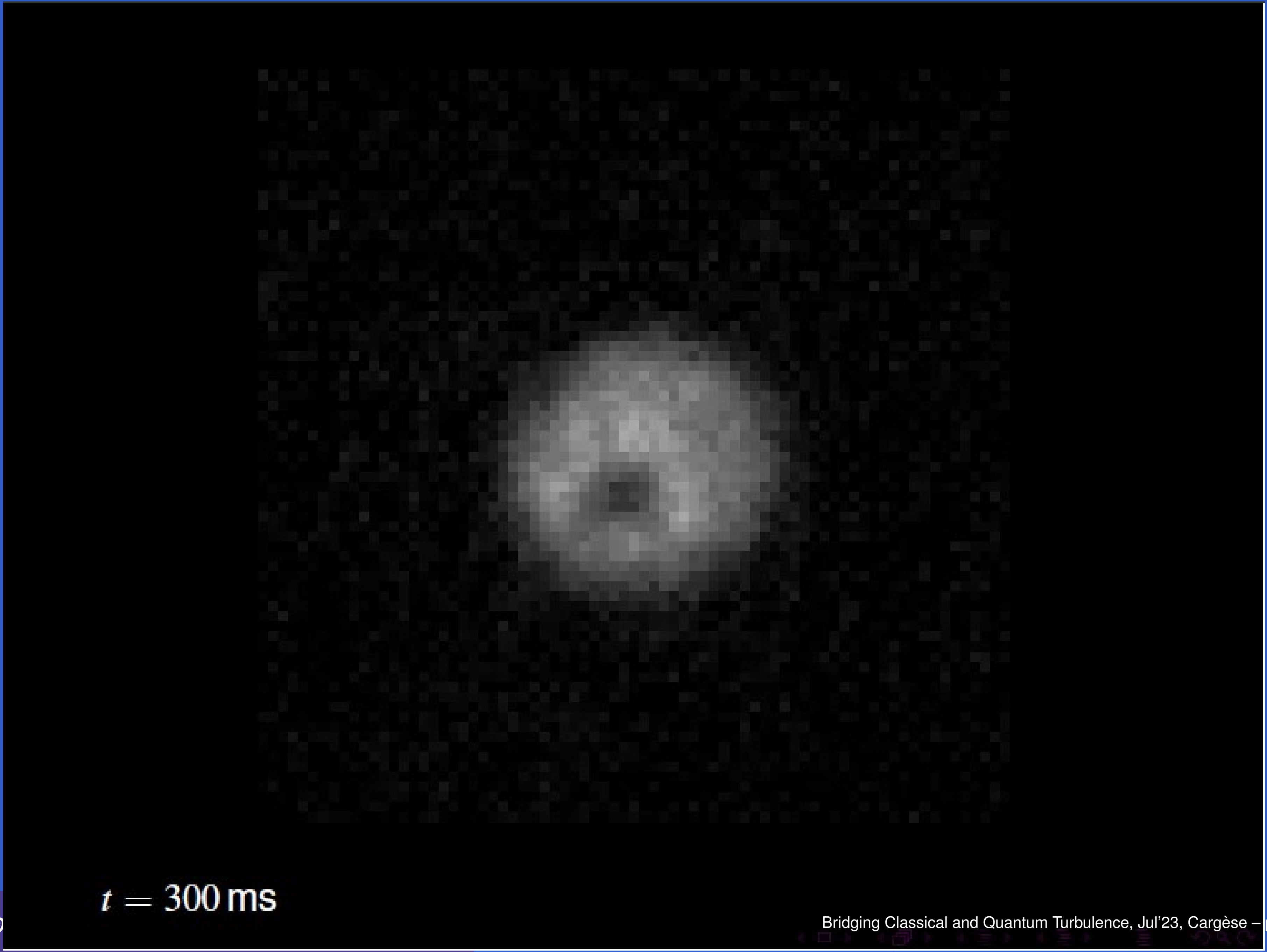

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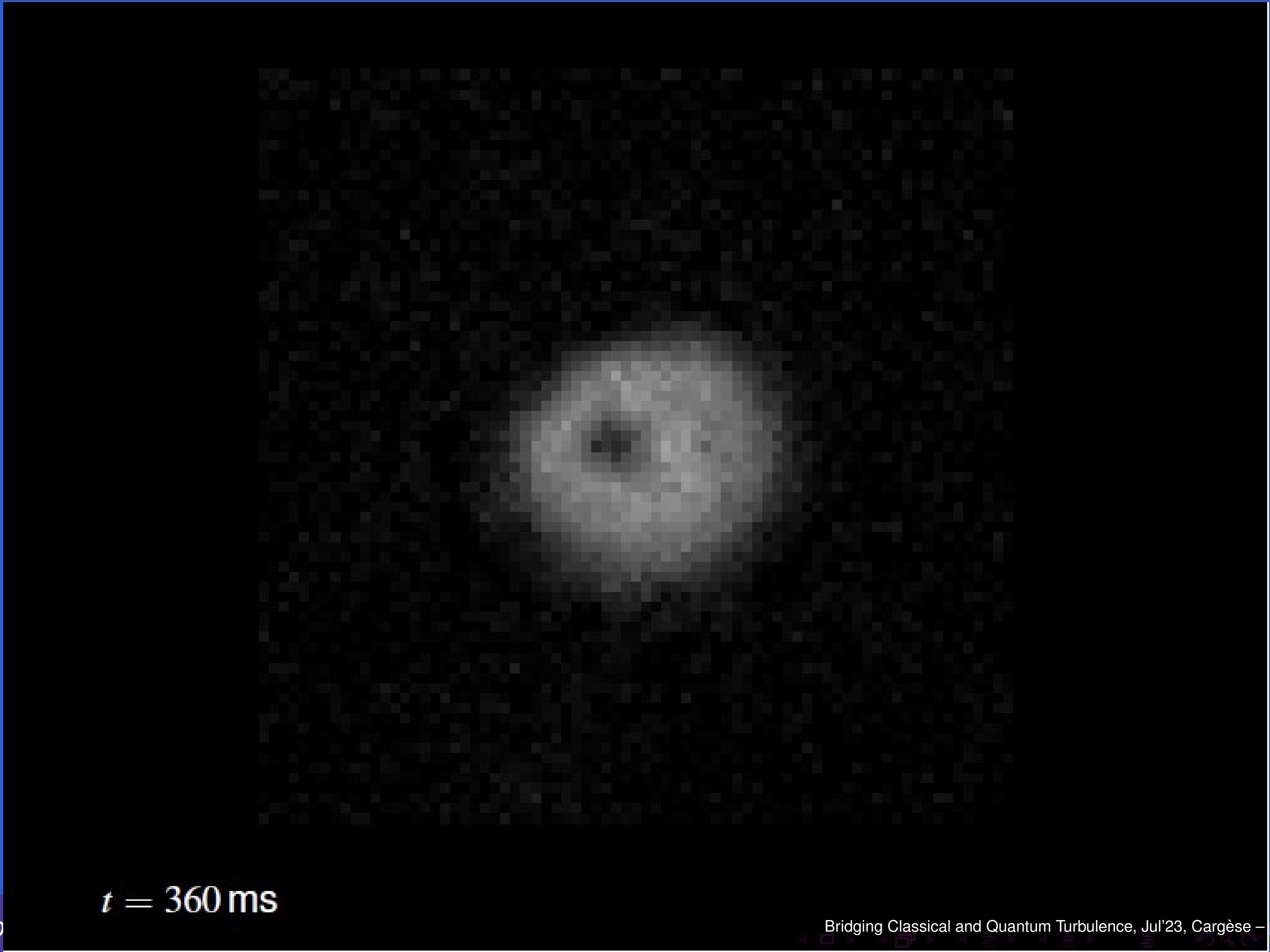

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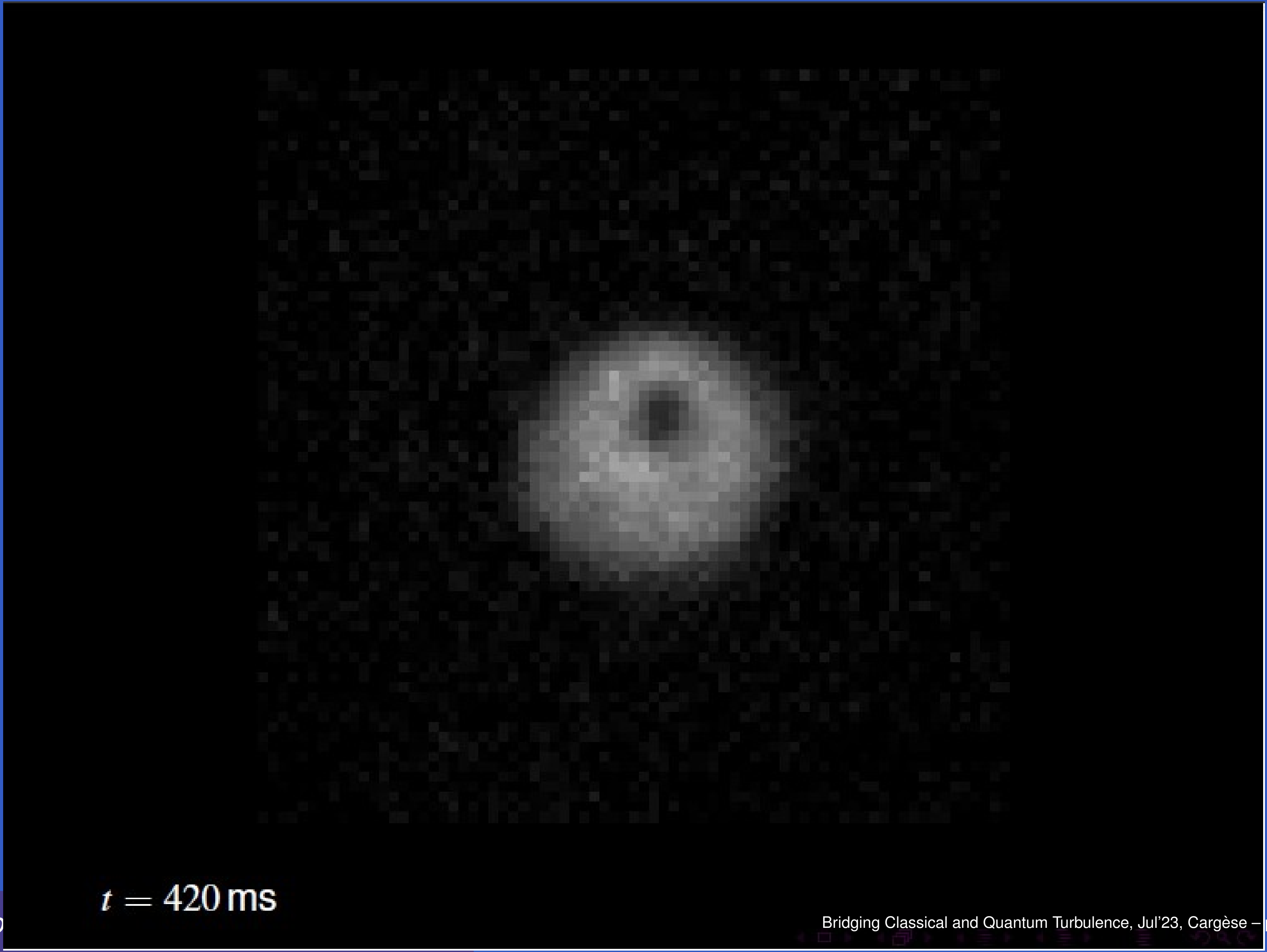

Static vortex dipole

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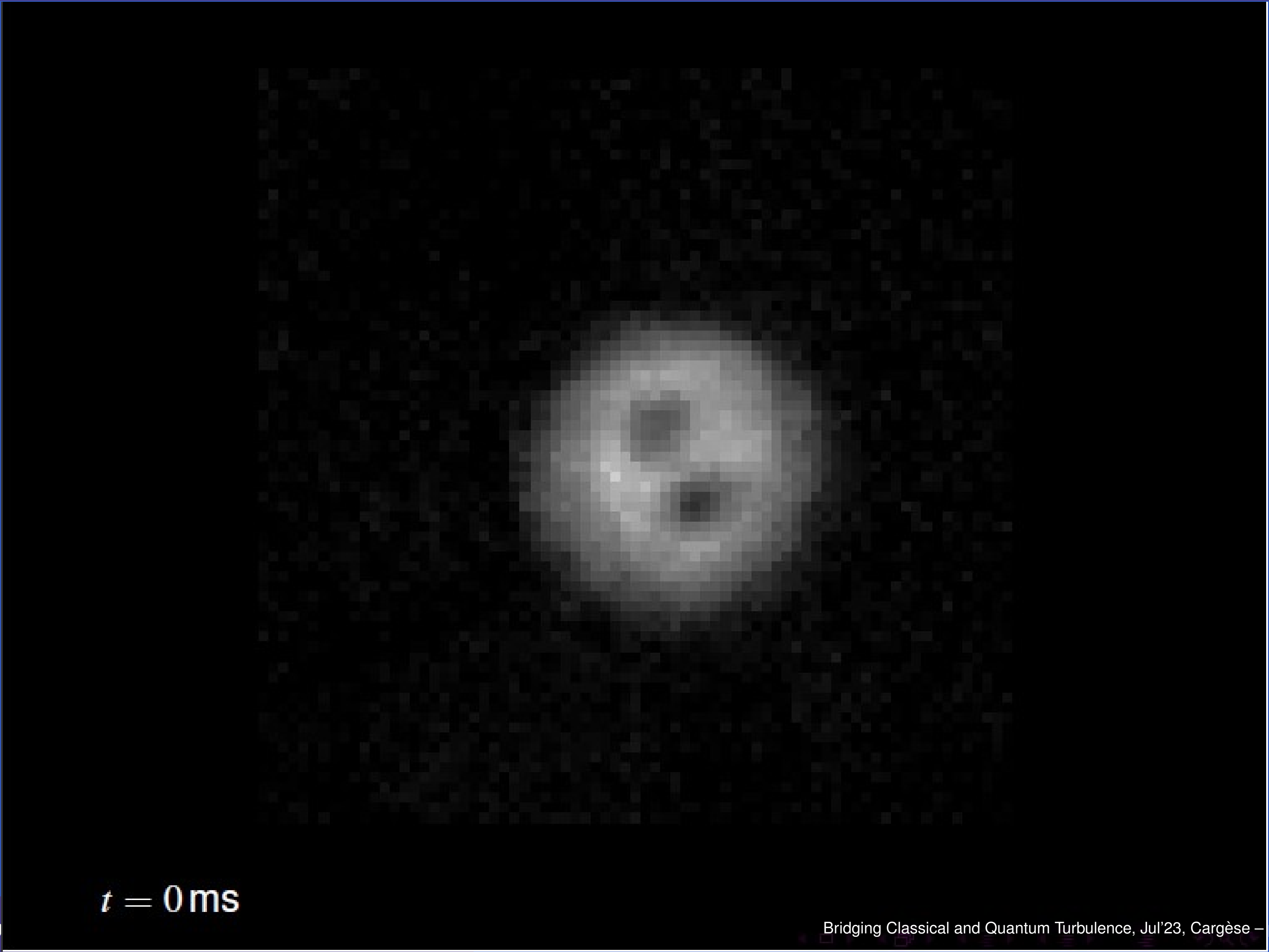

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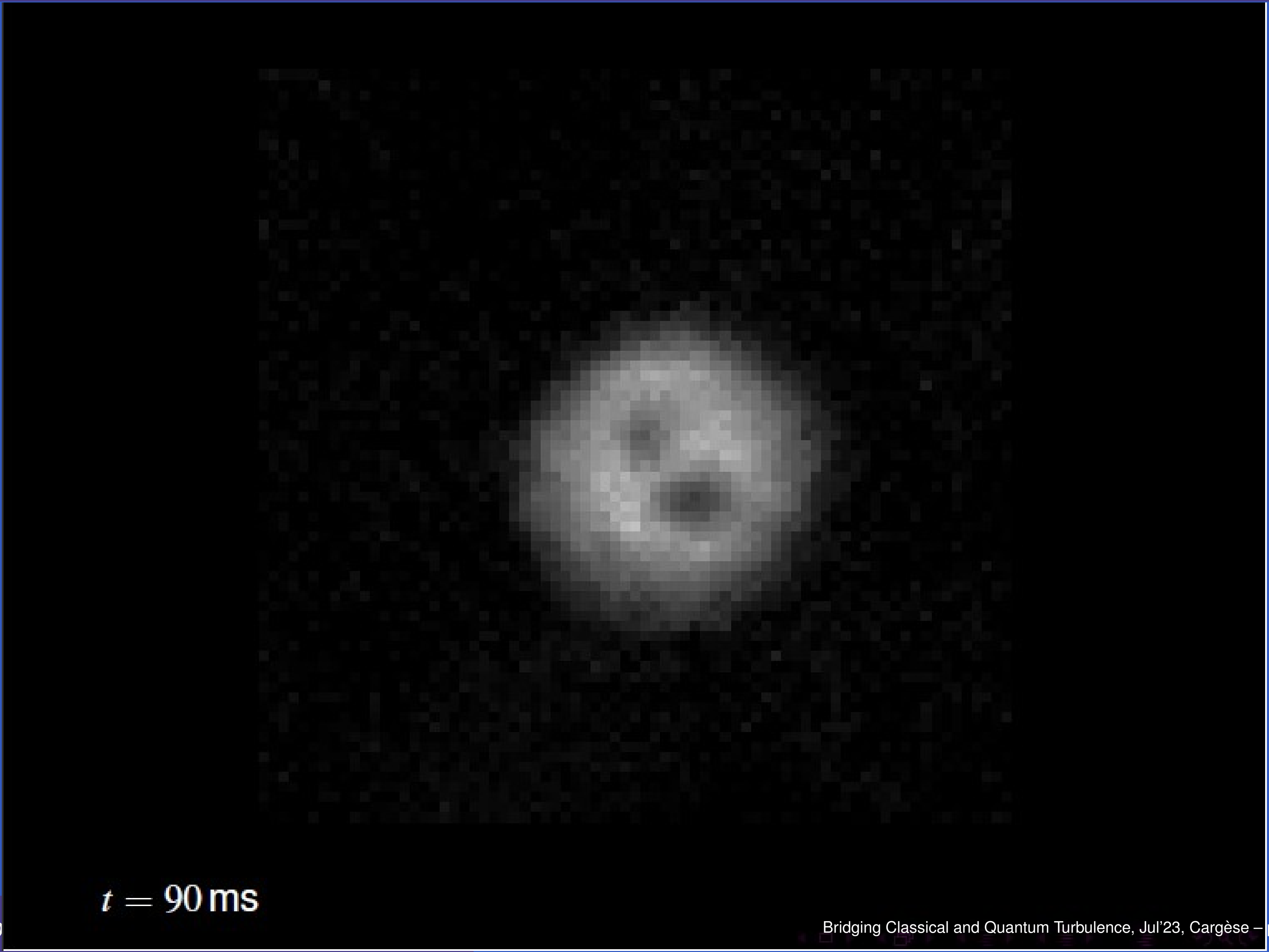

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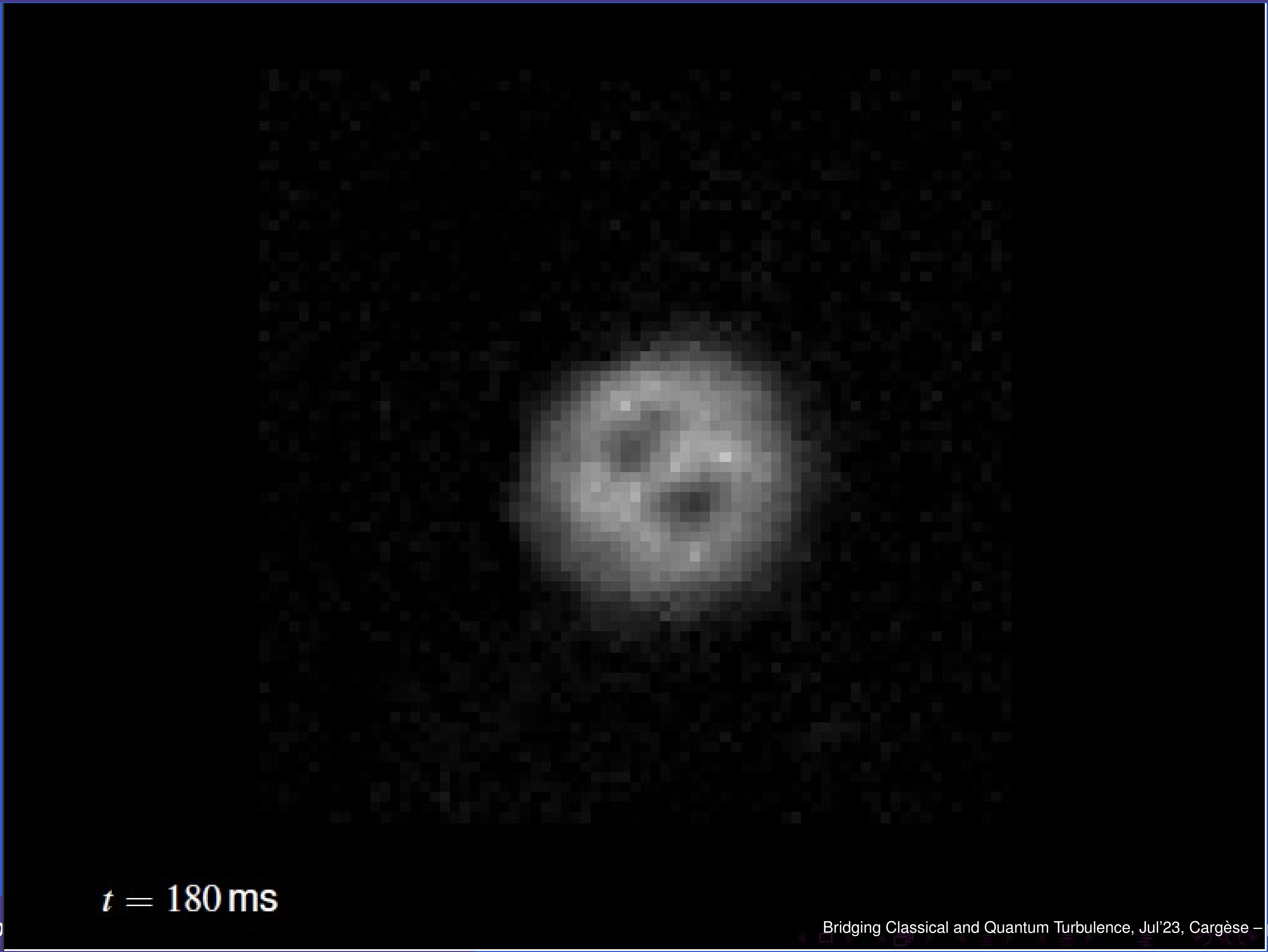

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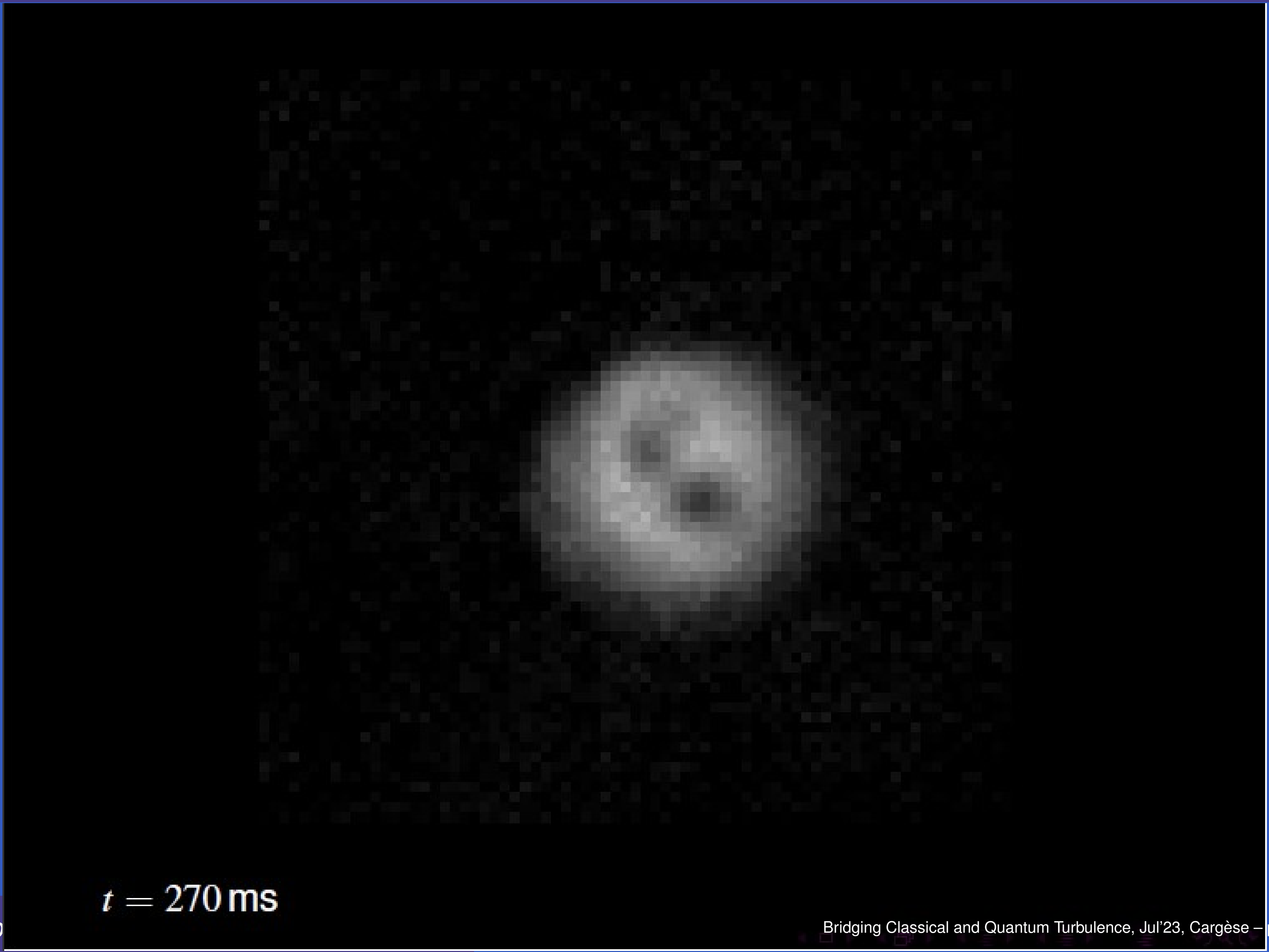

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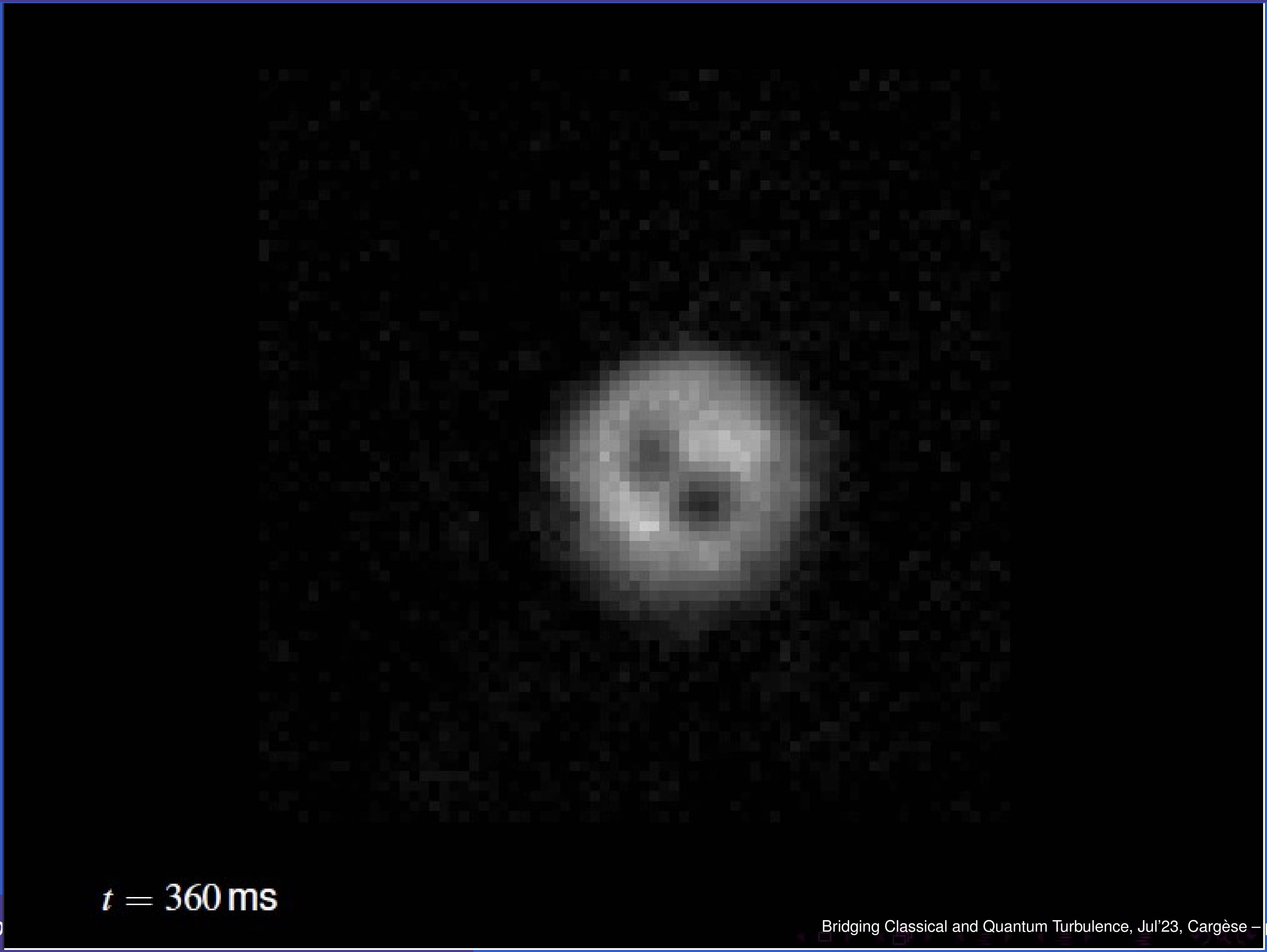

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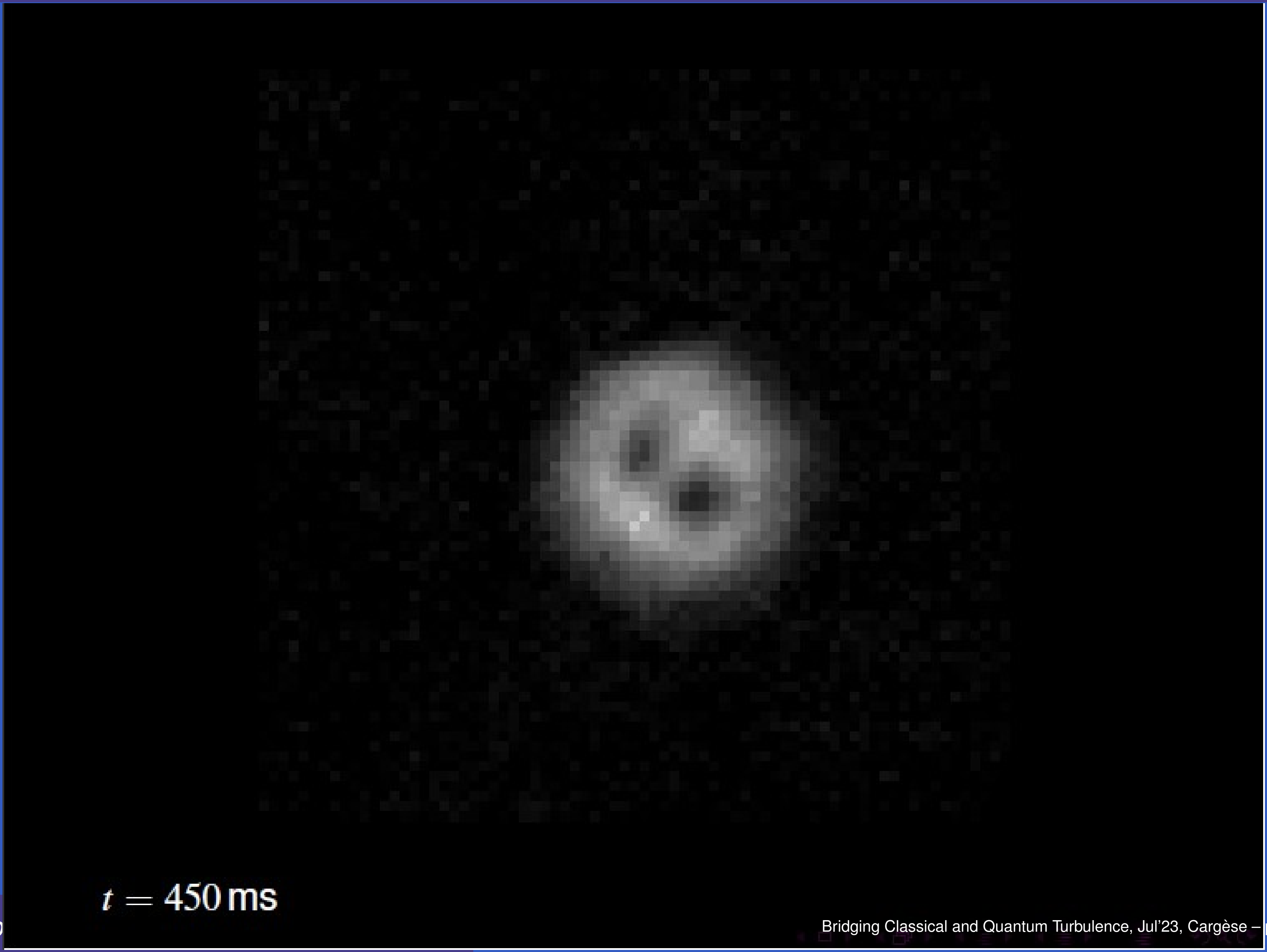

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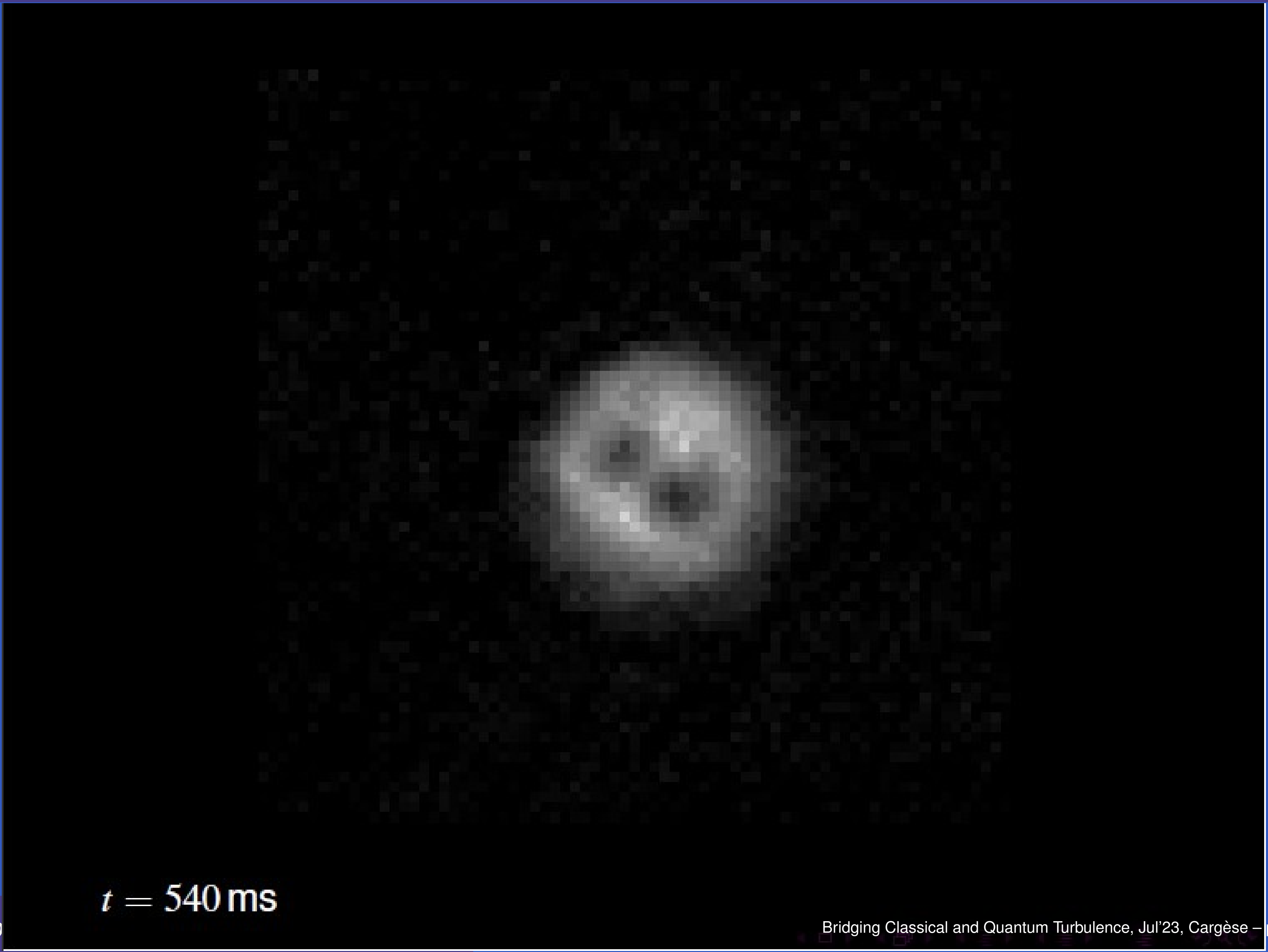

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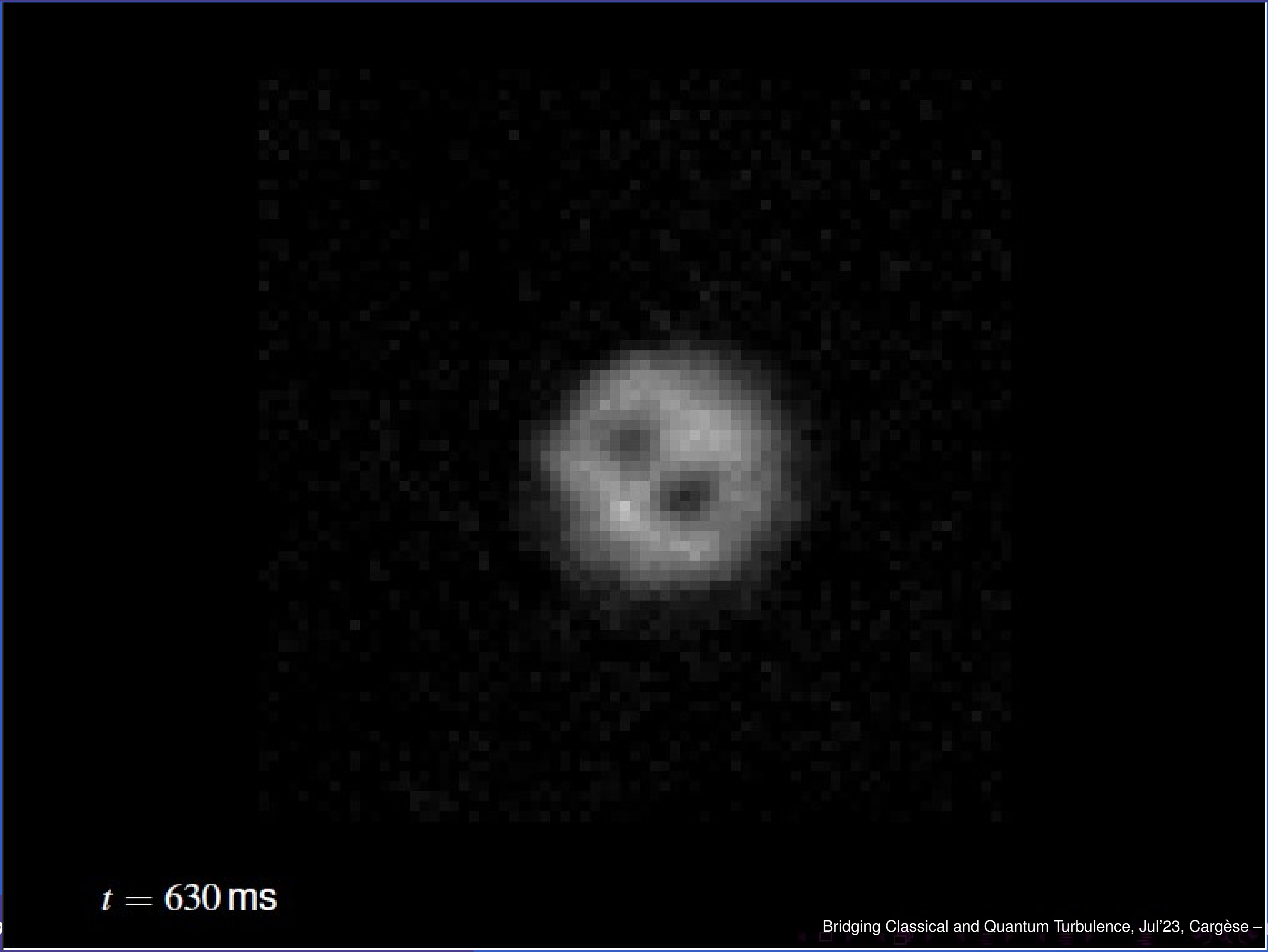

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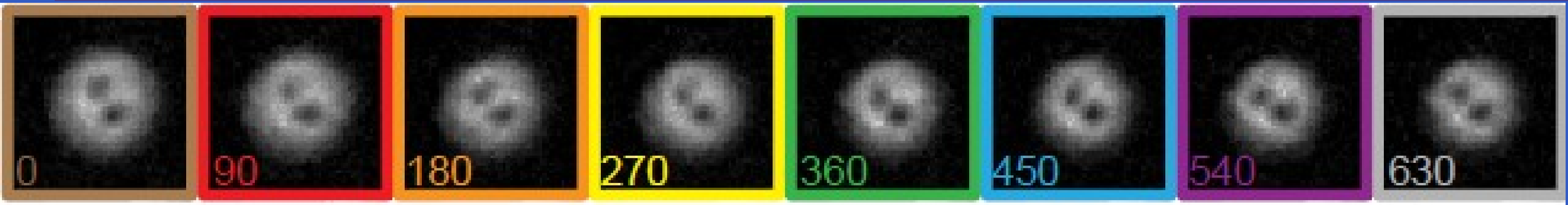
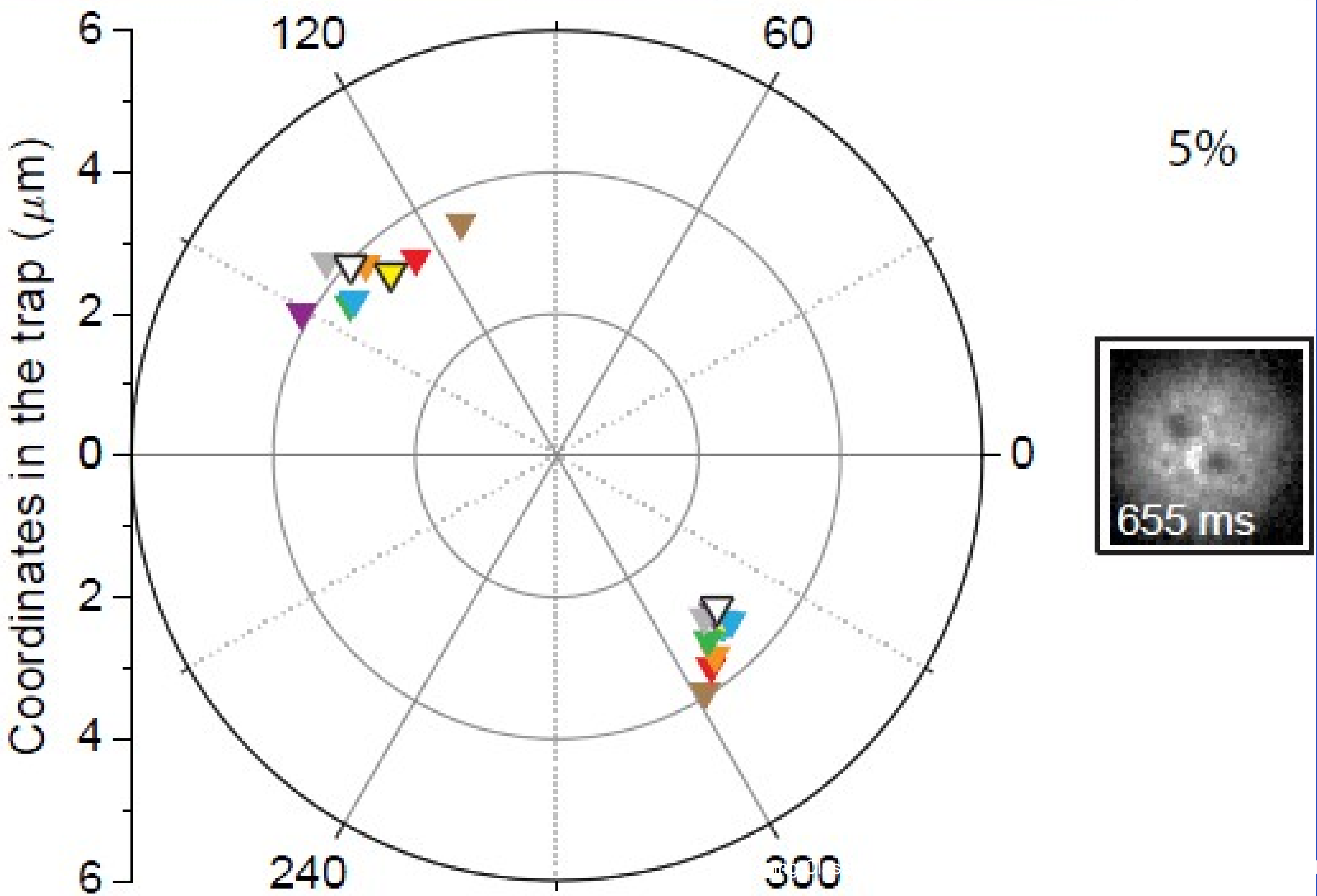
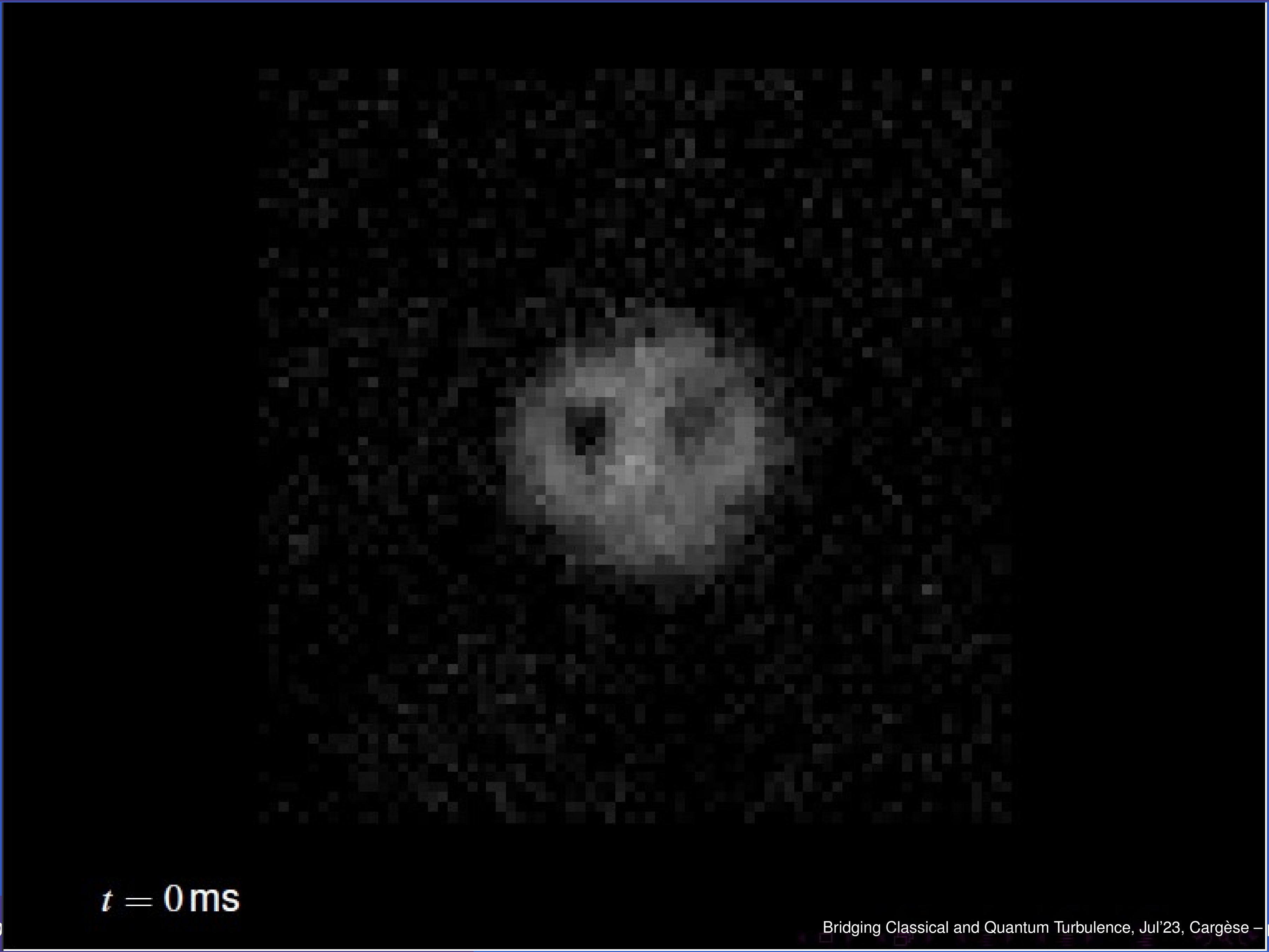




Periodic vortex dipole

Experimental BEC vortices: periodic cycle



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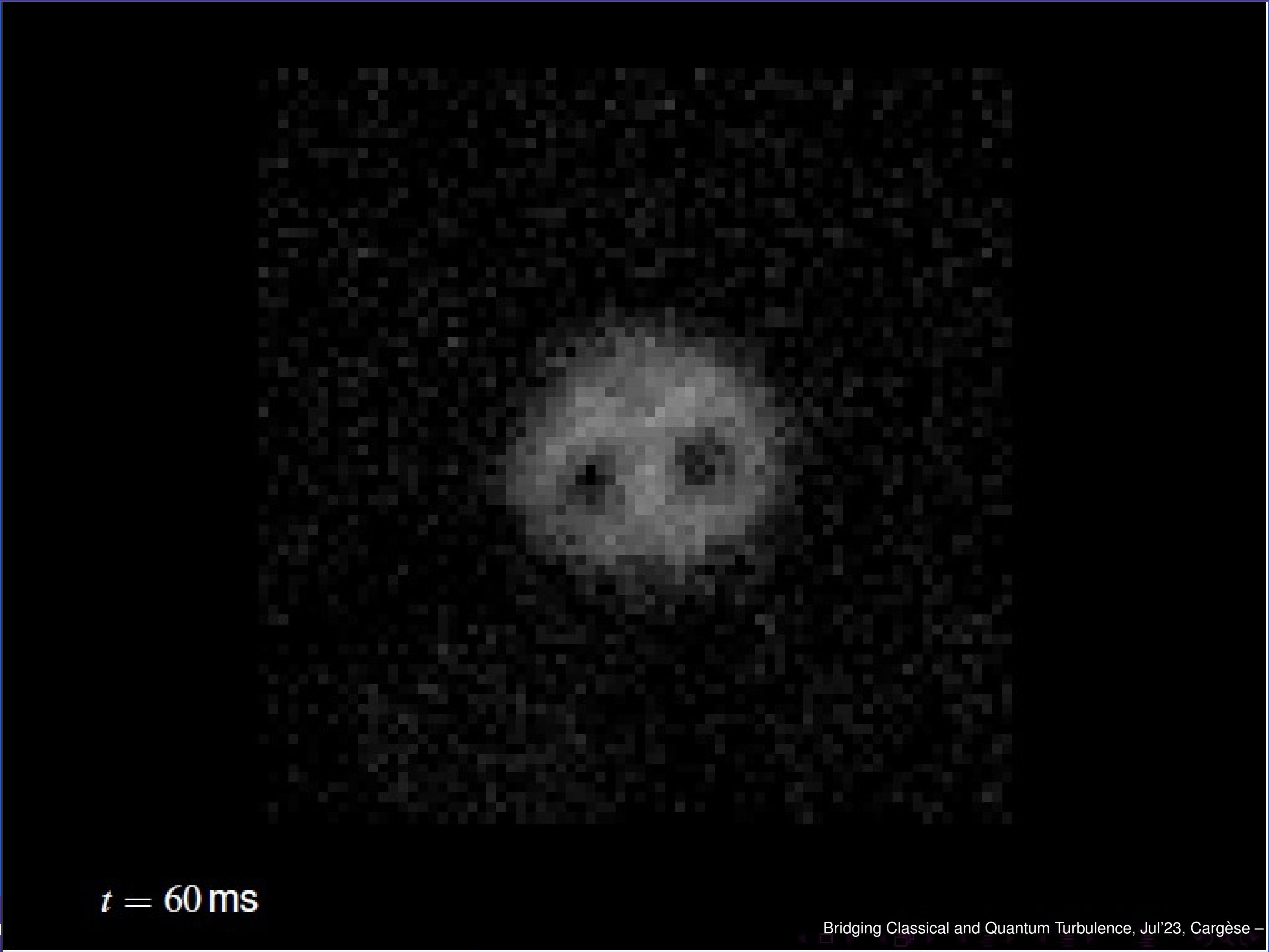

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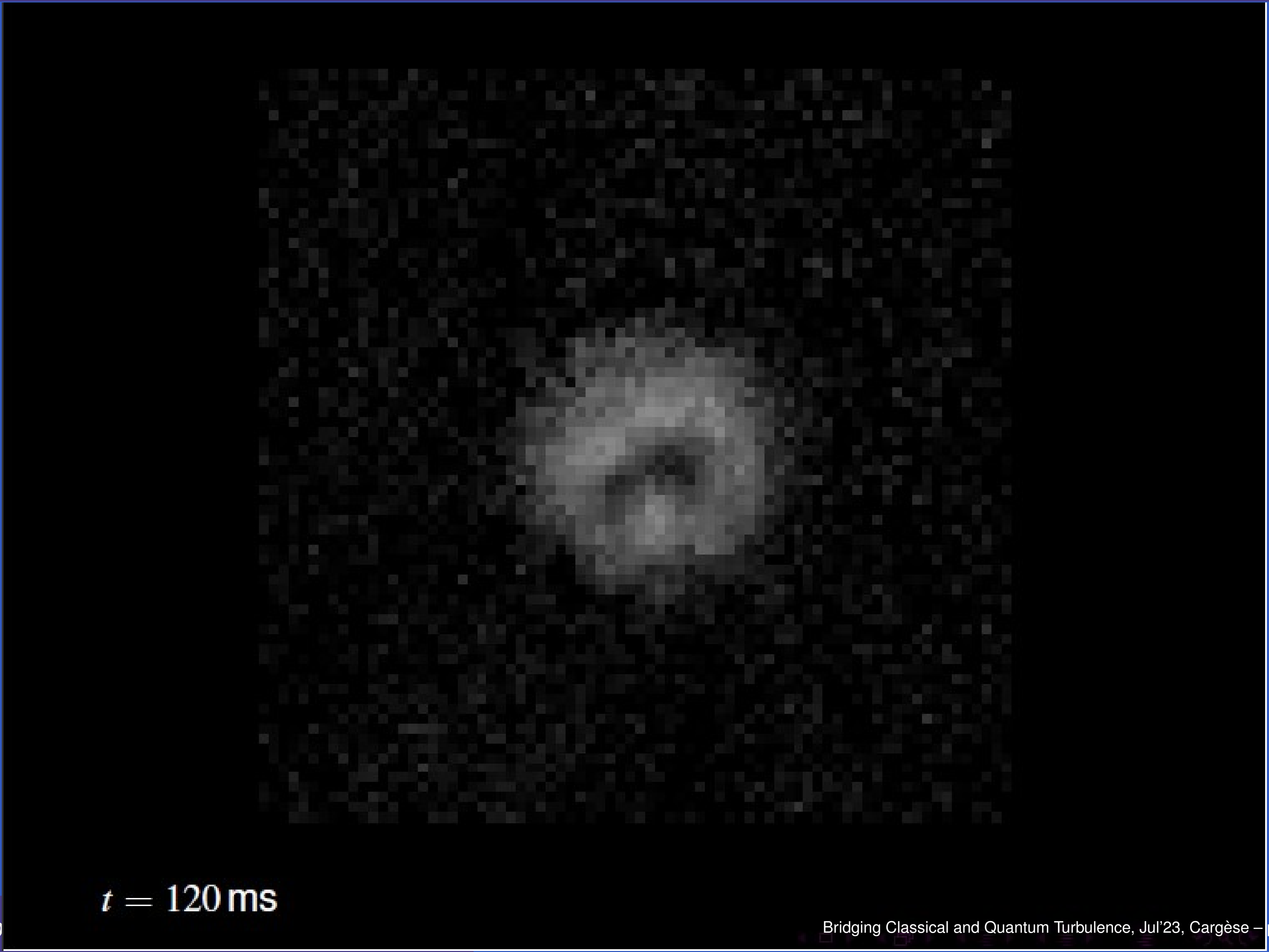

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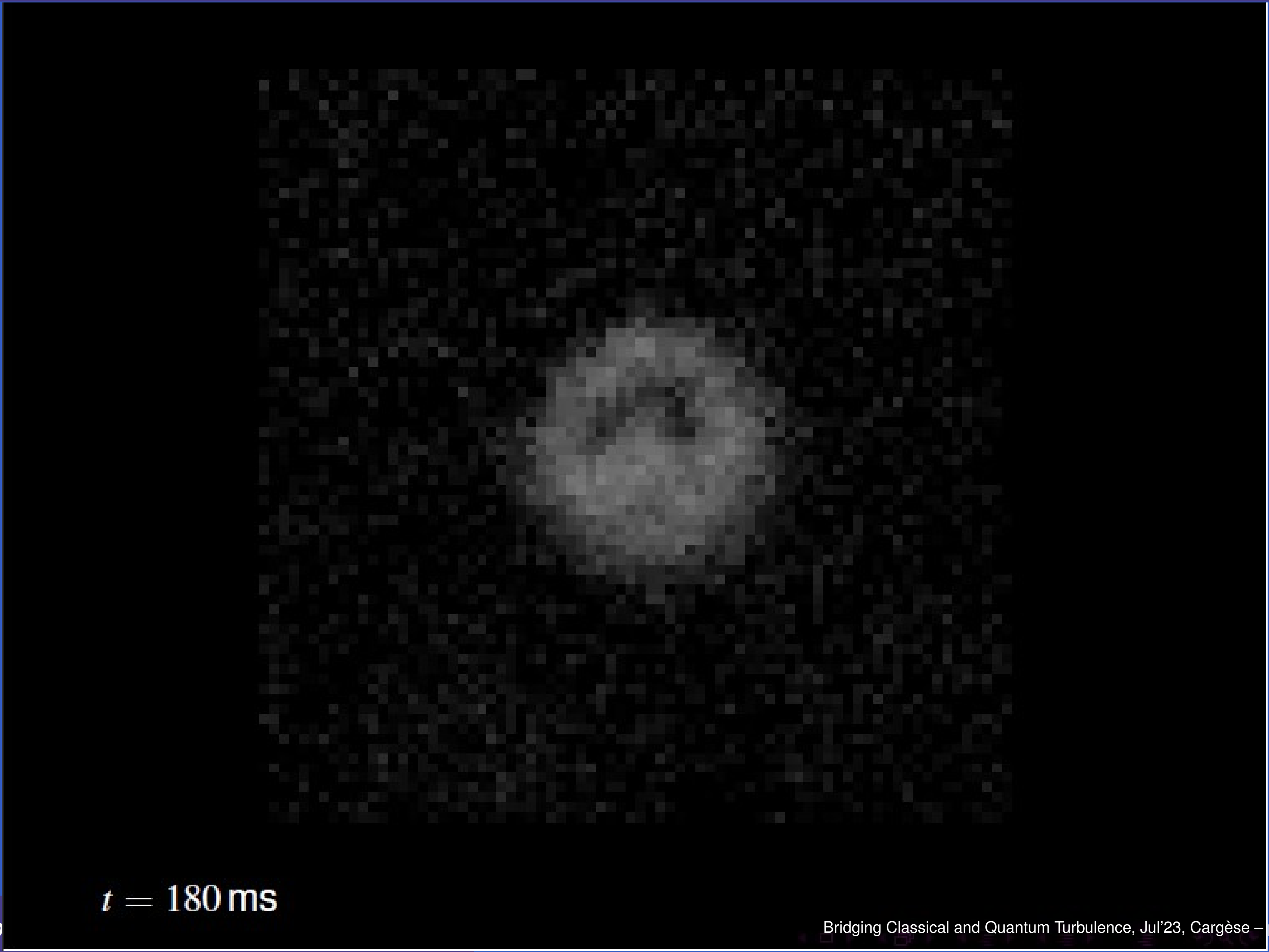

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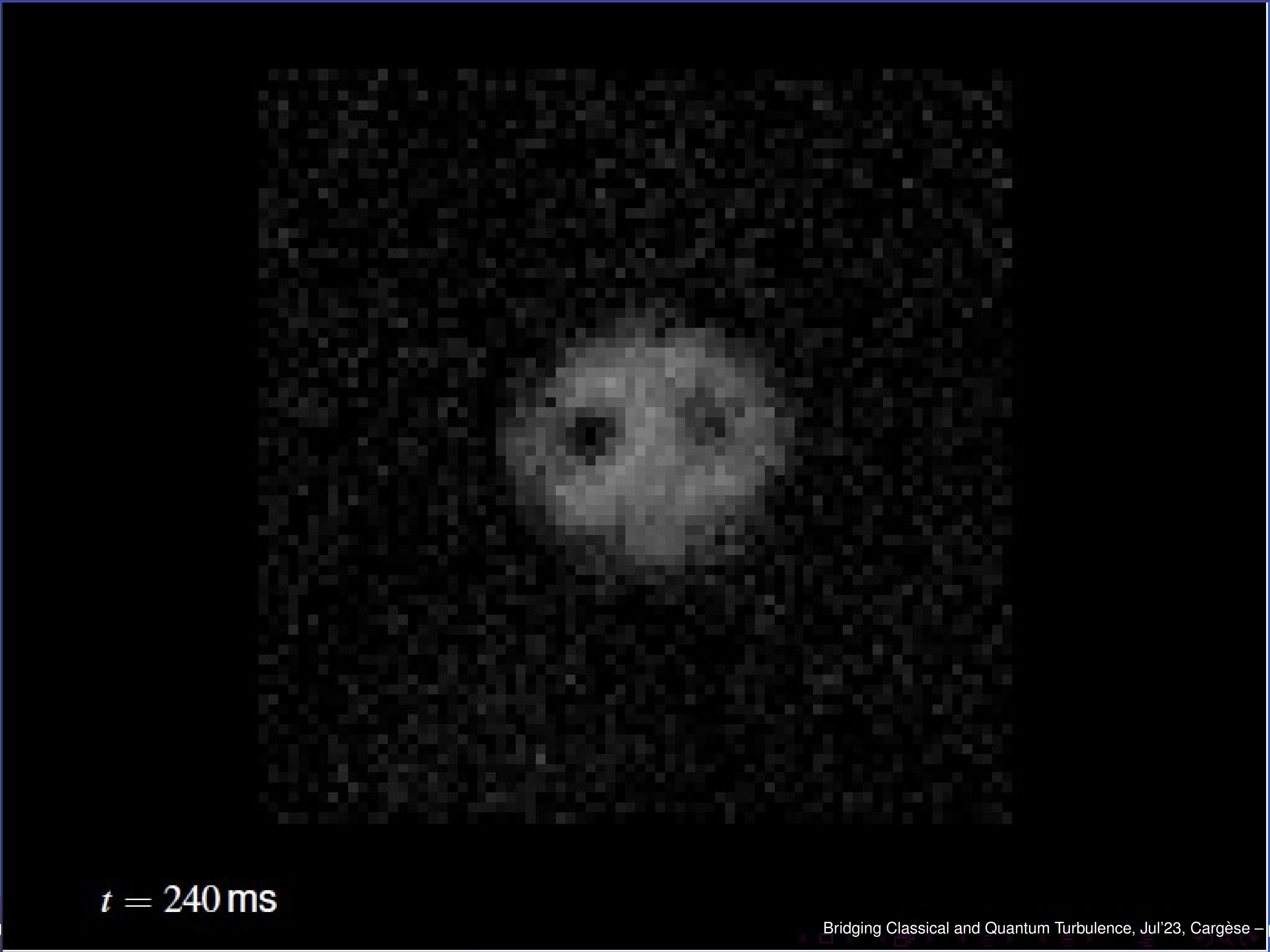

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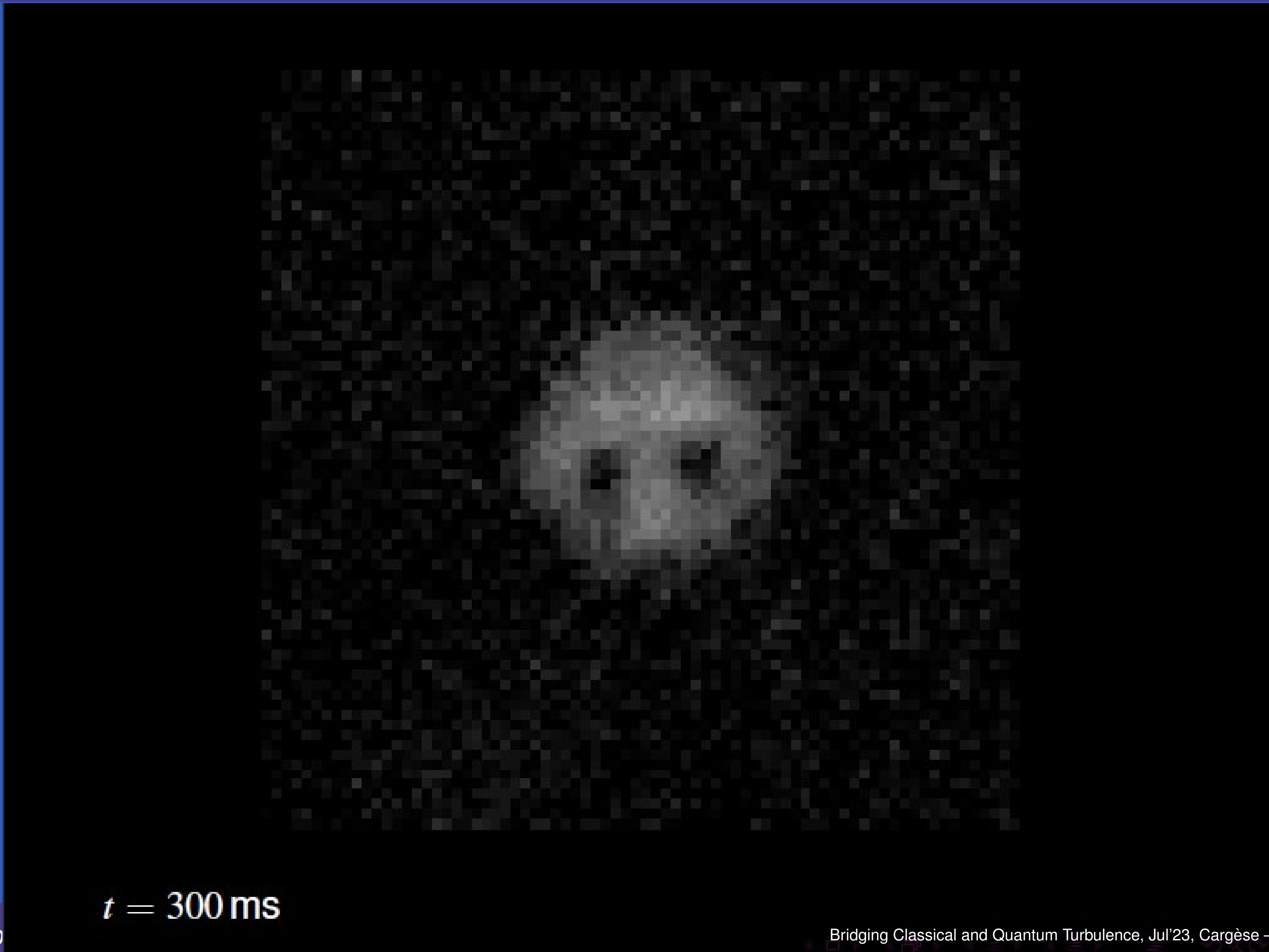

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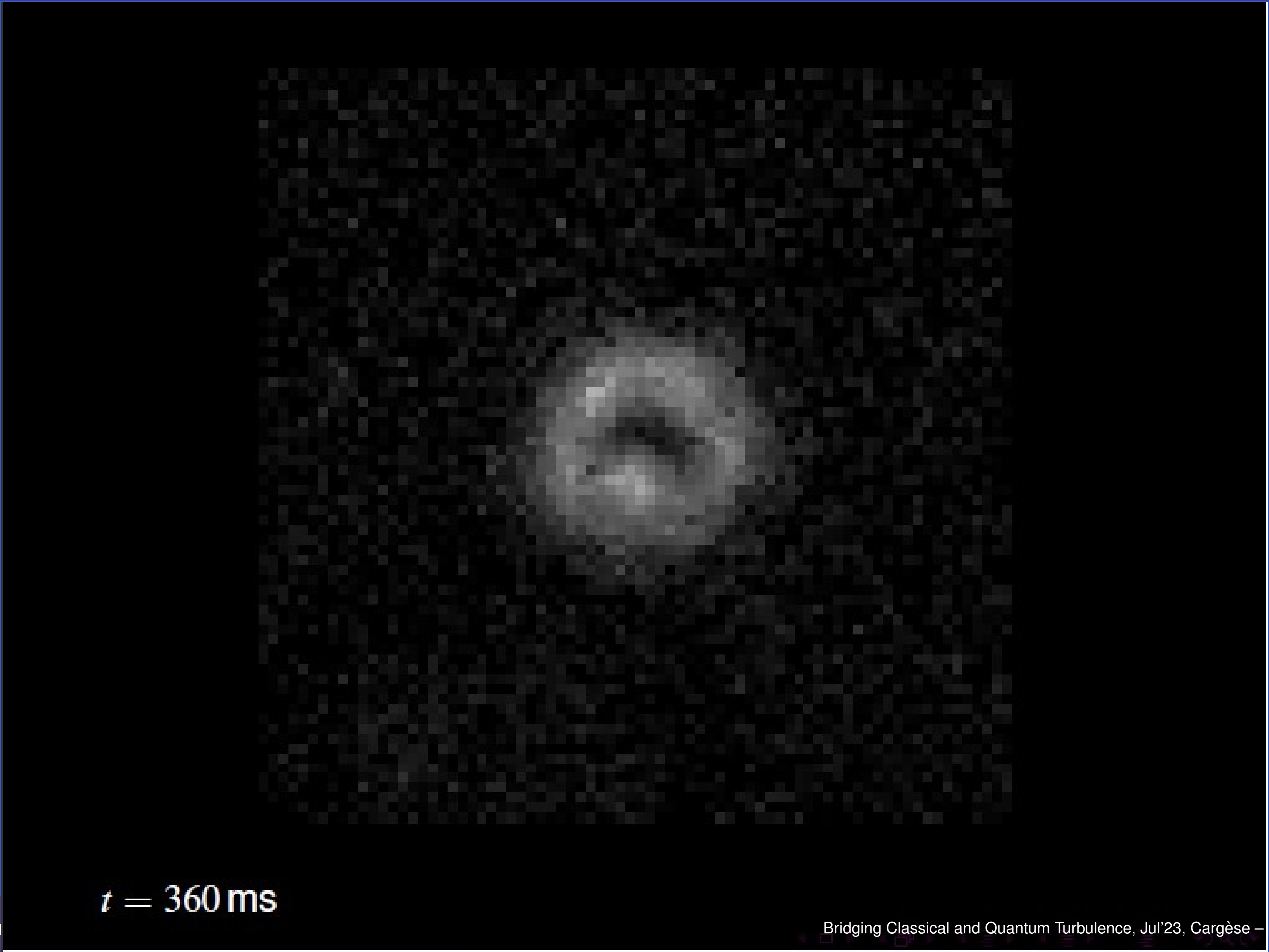

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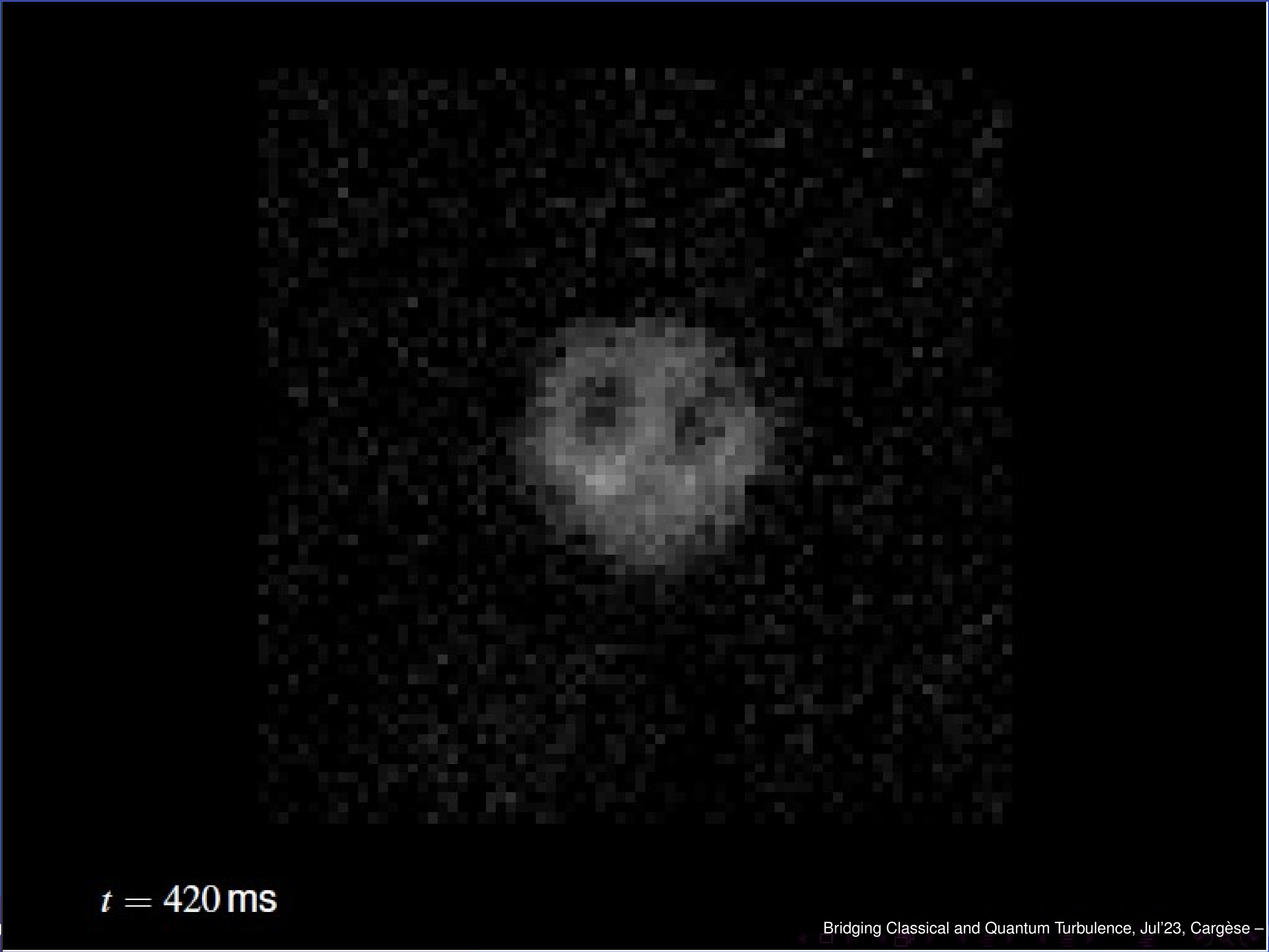

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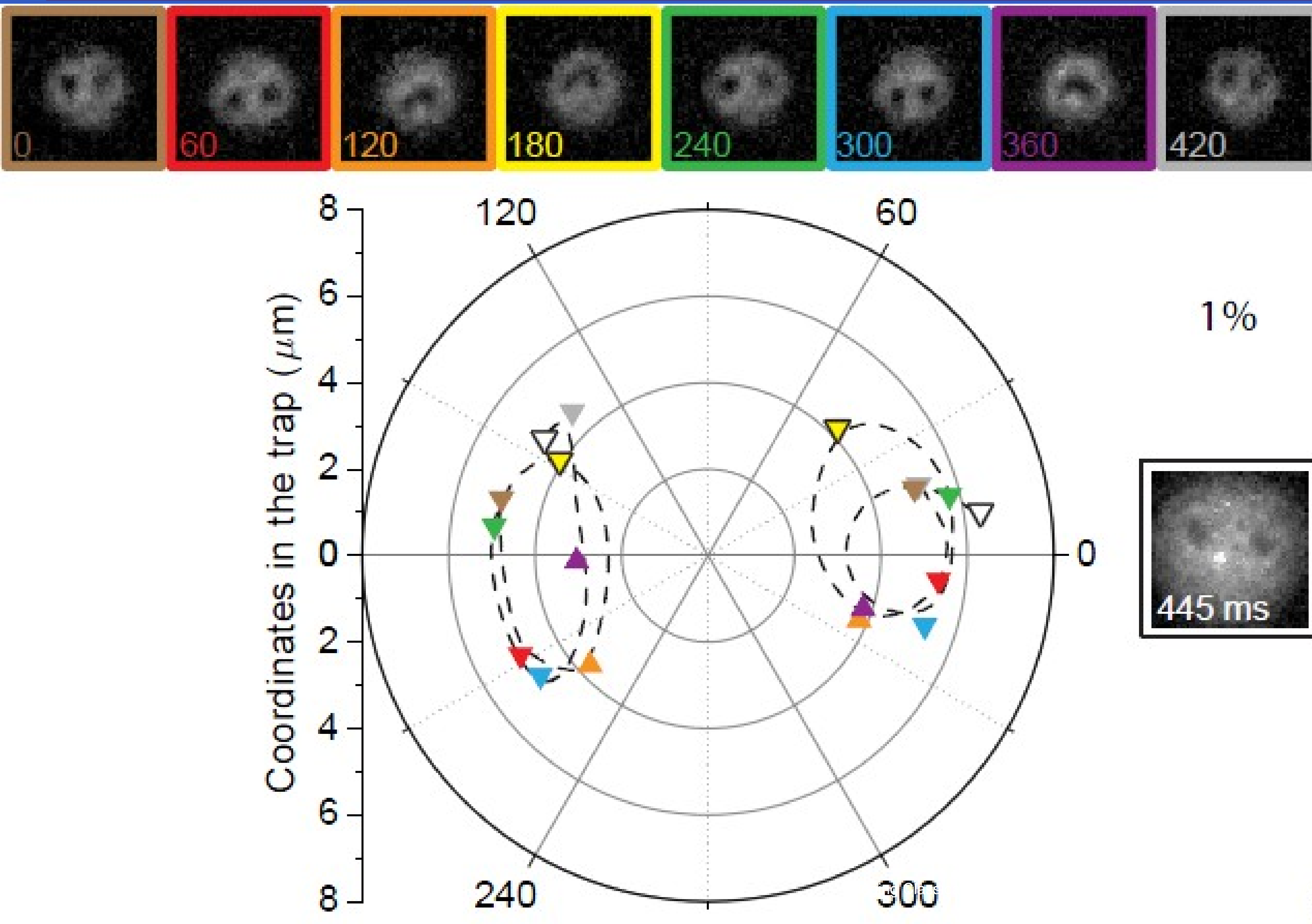
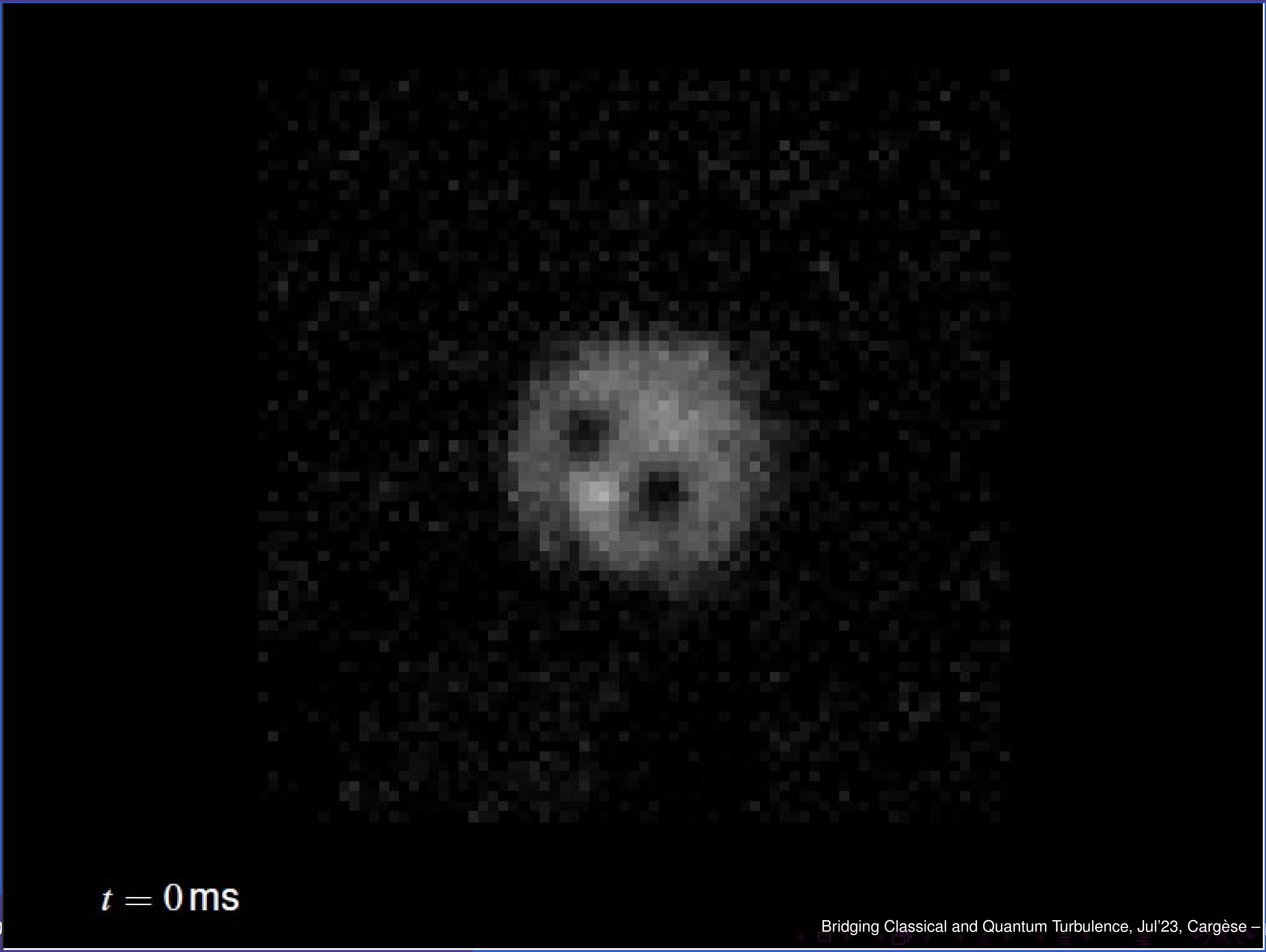


Vortex Spirograph

Experimental BEC vortices: epitrochoidal motion



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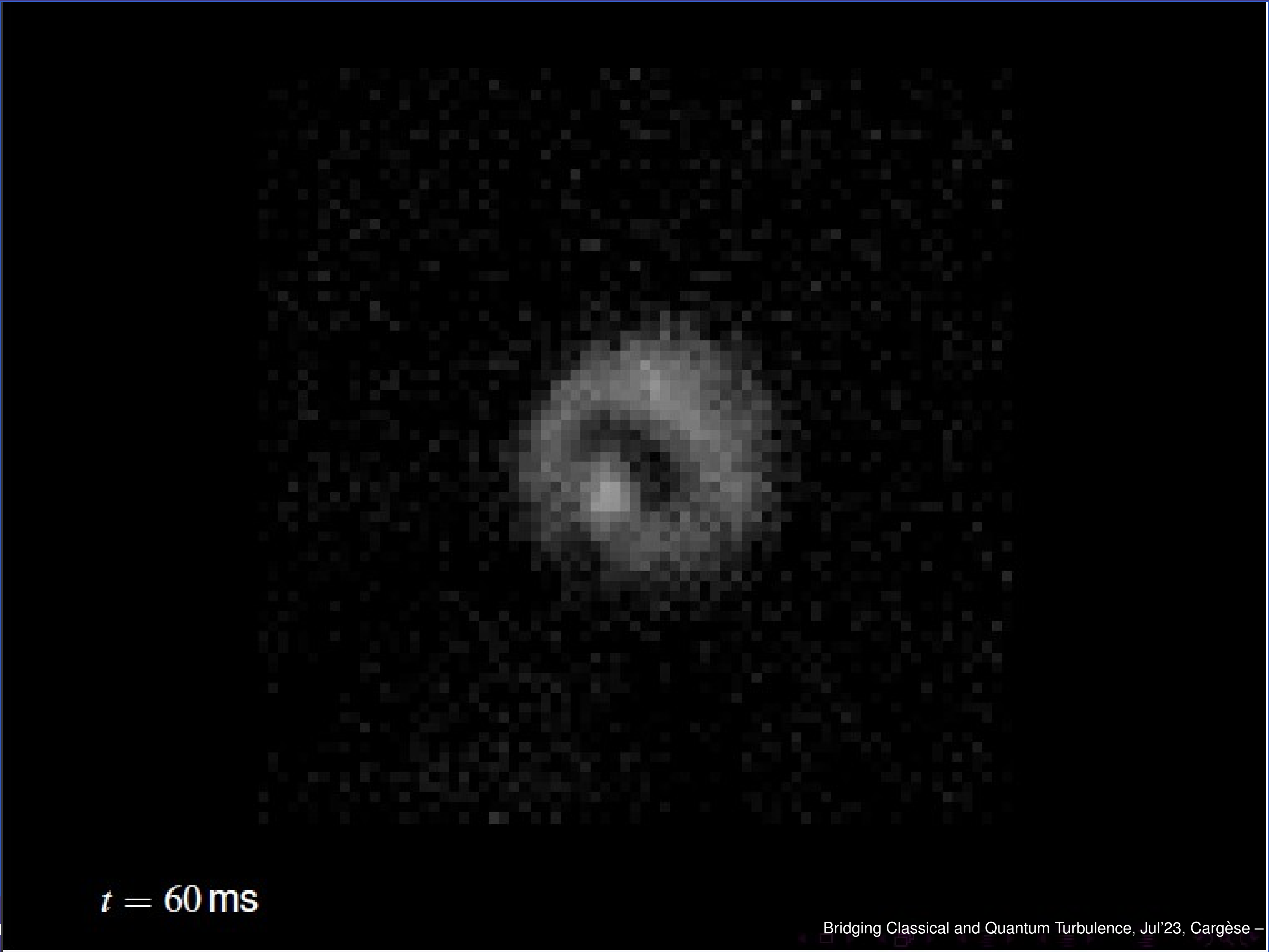

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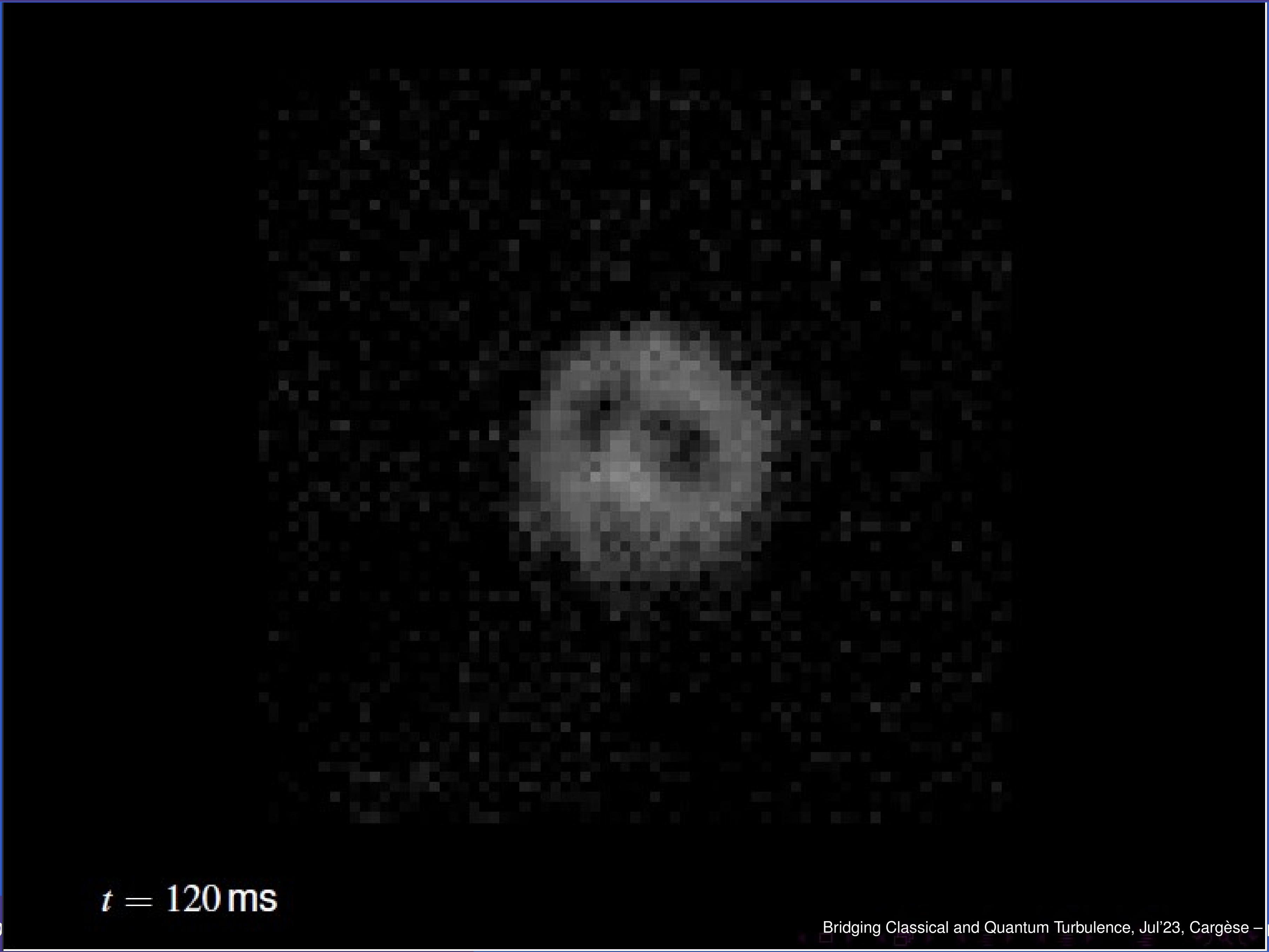

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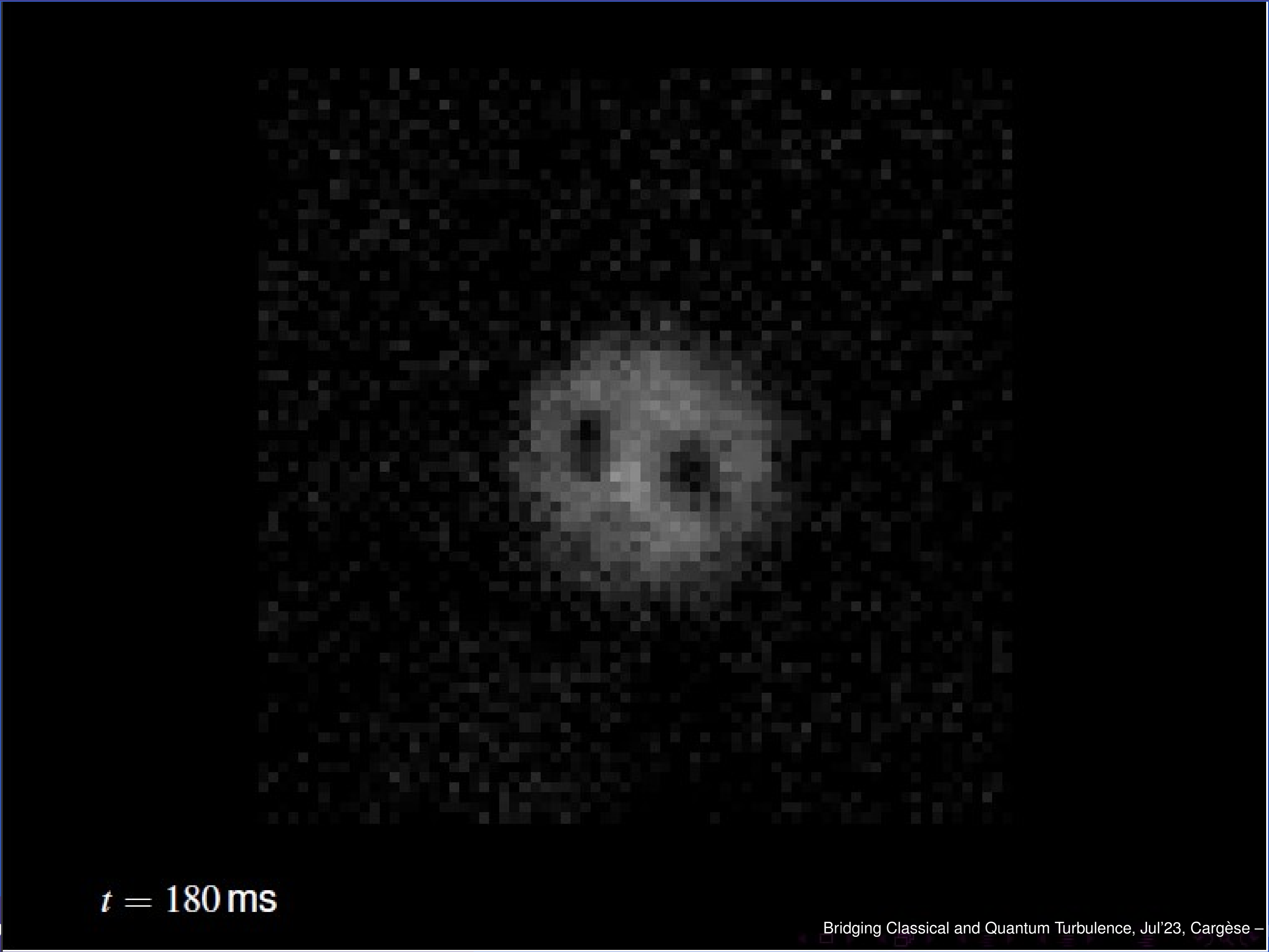

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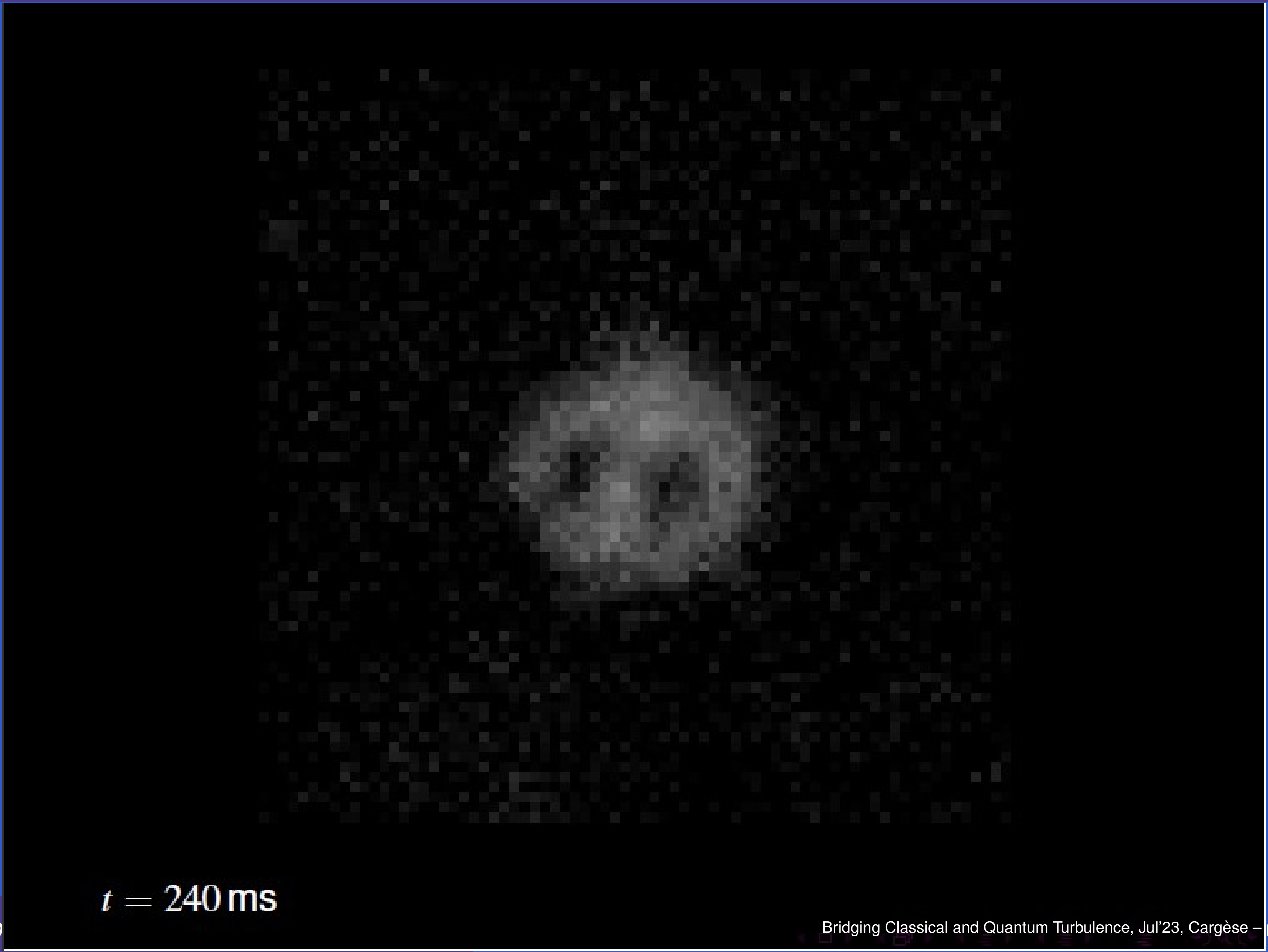

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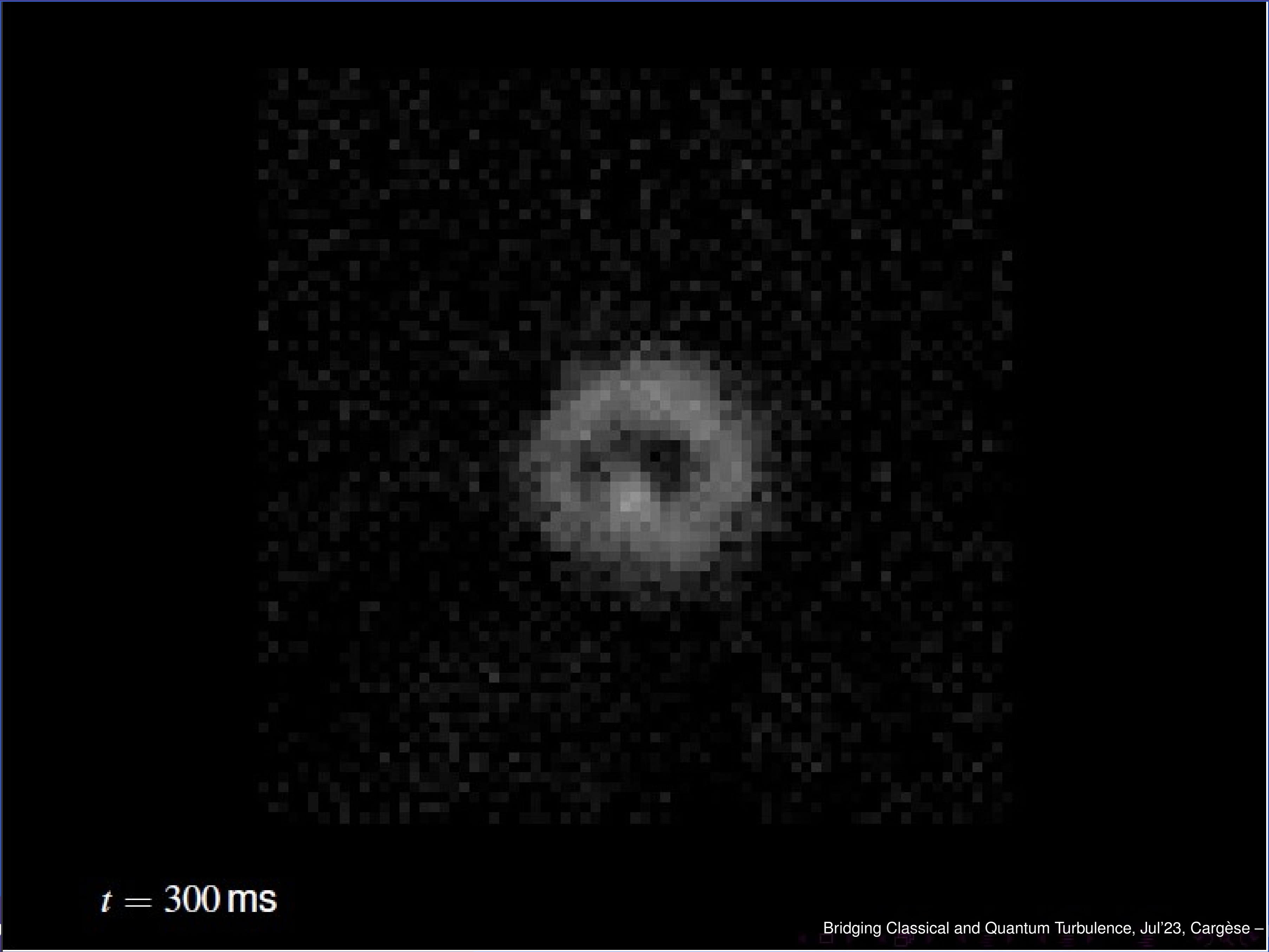

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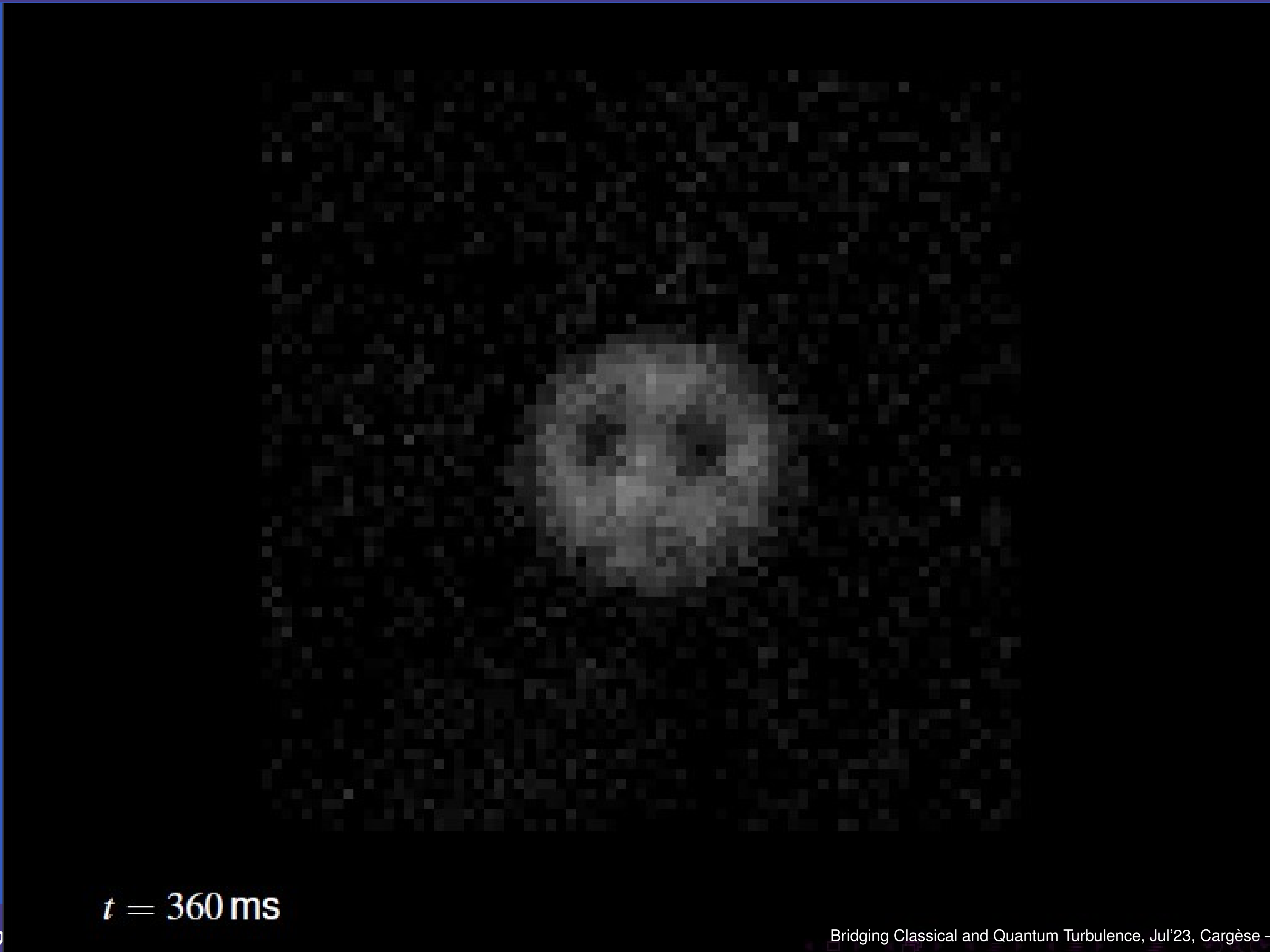

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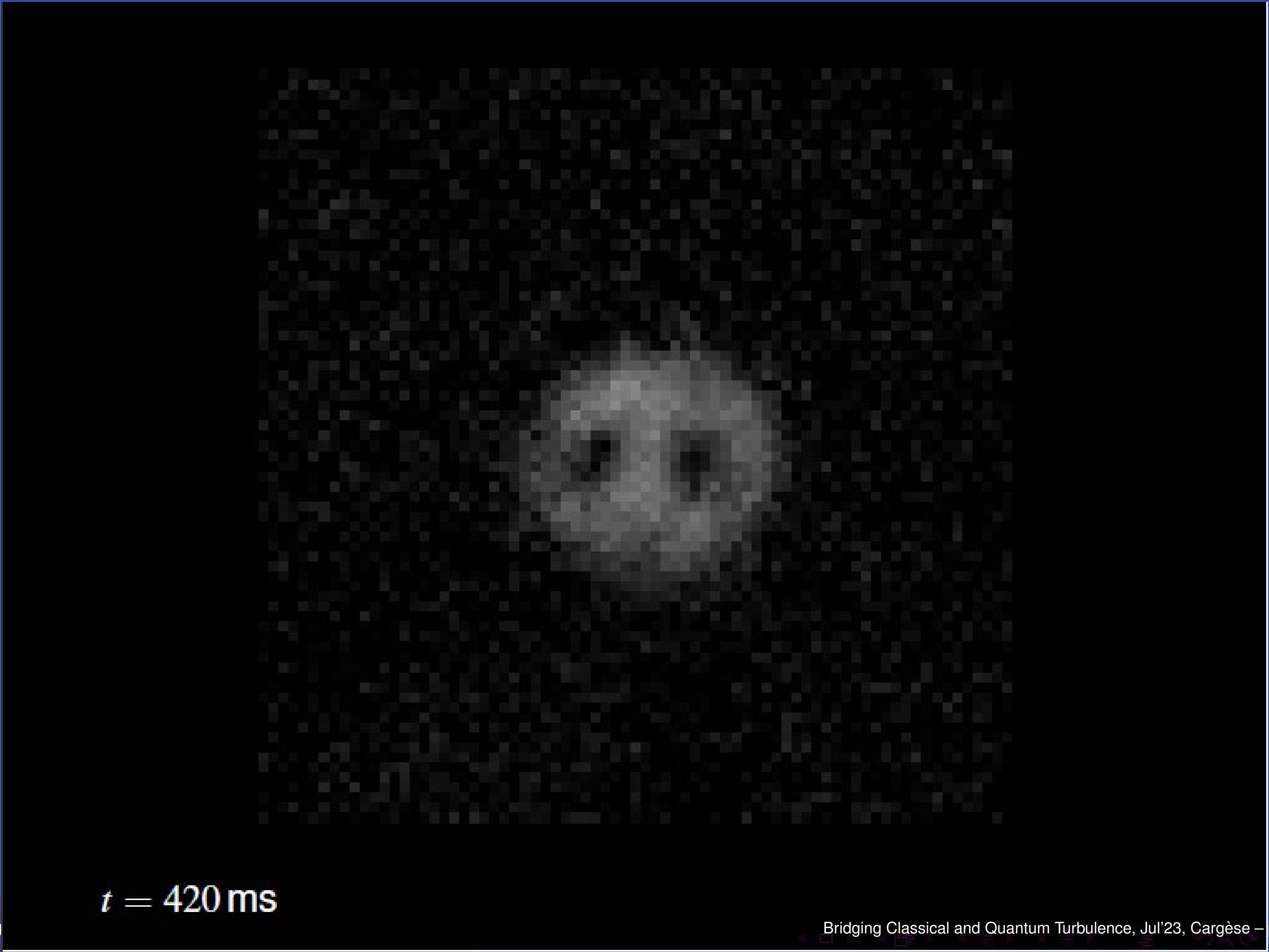

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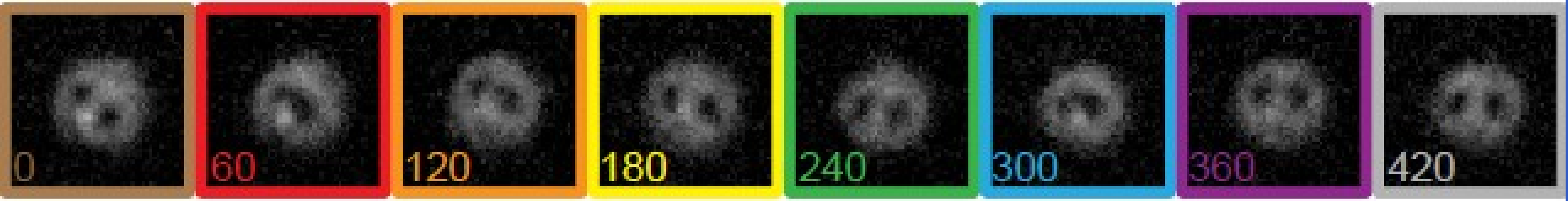
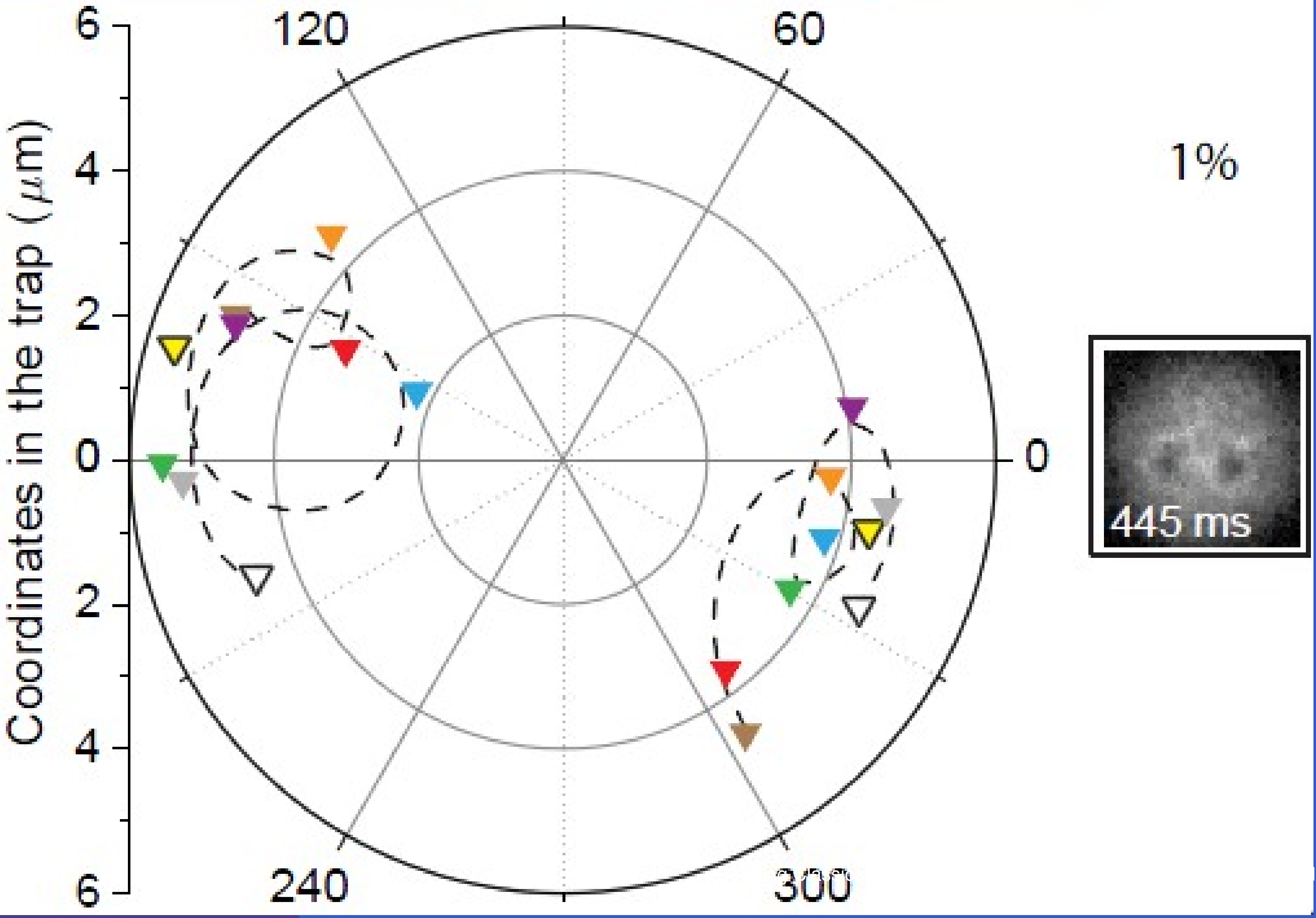




Vortex dynamics and their interactions in parabolically trapped BECs

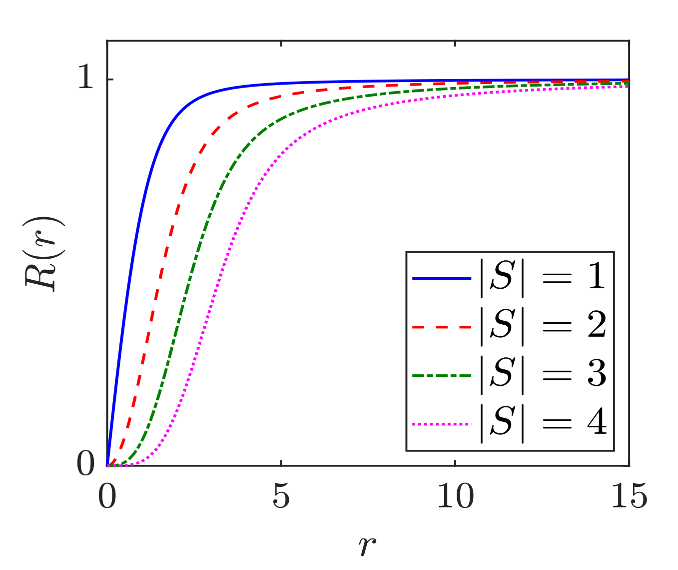
Vortex: a single one without external potential

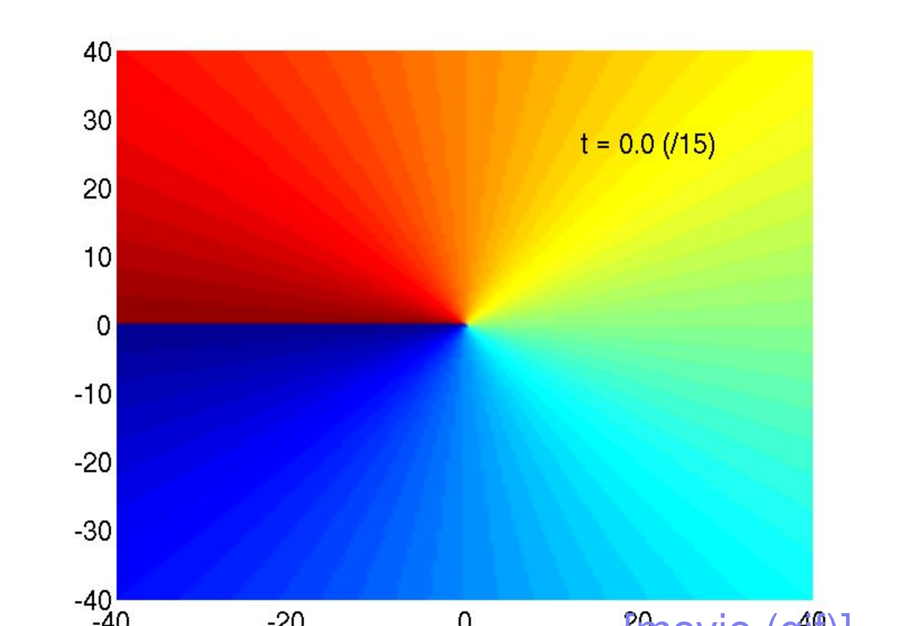
- Take solution with topological charge S:

$$u(x,y,t) = R(r)\, e^{iS\theta}\, e^{-i\mu t}$$

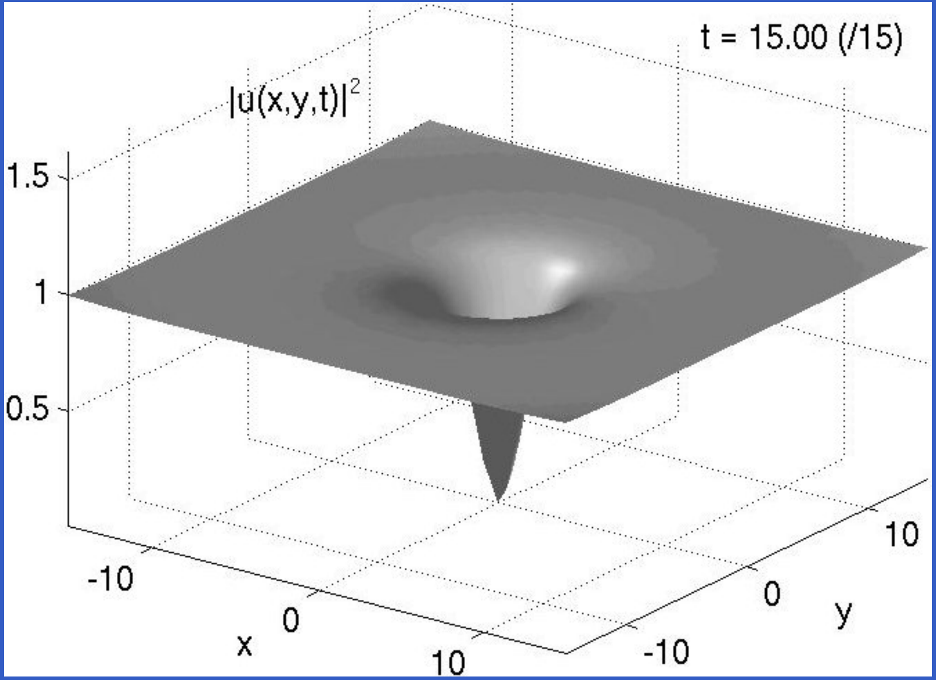
- Vortex radial profile satisfies:

$$\left(\mu - \frac{S^2}{2r^2}\right) R + \frac{1}{2r} R' + \frac{1}{2} R'' - R^3 = 0$$





[movie (gif)]



Vortex: precession in the magnetic trap (MT)

- Precession frequency for a vortex inside a MT ($V(r) = \frac{1}{2}\Omega^2 r^2$) at distance r from center.

$$S\,\omega_{\rm pr} = \frac{\omega_{\rm pr}^0}{1 - \left(\dfrac{r}{R_{TF}}\right)^2}$$

- Precession frequency close to center: $\omega_{\rm pr} \approx \omega_{\rm pr}^0 \left[= \dfrac{\Omega^2}{2\mu} \ln\left(A\dfrac{\mu}{\Omega}\right)\right]$ ($A \approx 8.88$).

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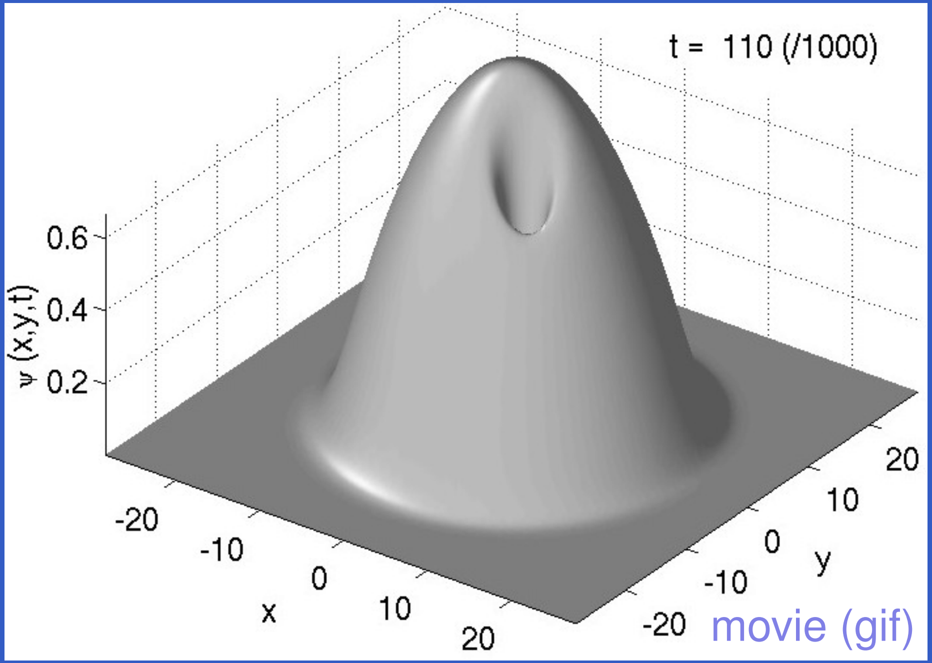
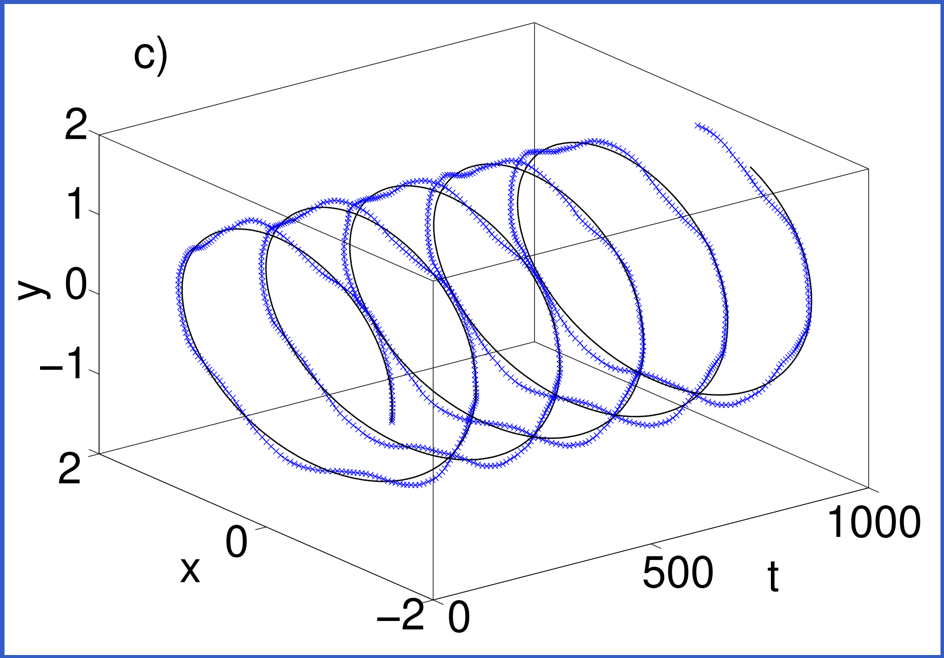
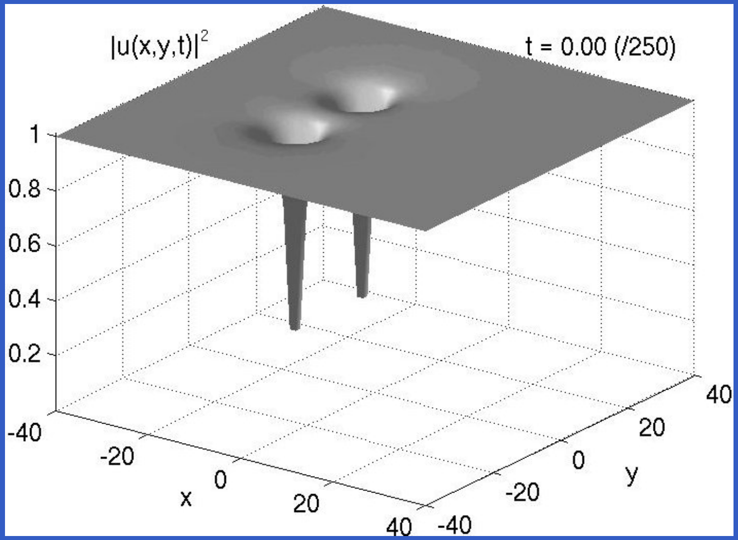


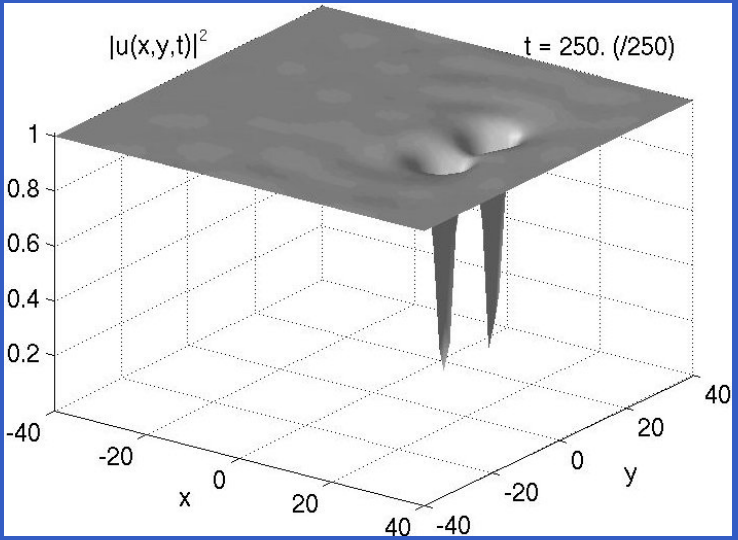


Vortex interactions: pairwise dynamics

- Opposite charge (vortex dipole) $\rightarrow$ run parallel to each other

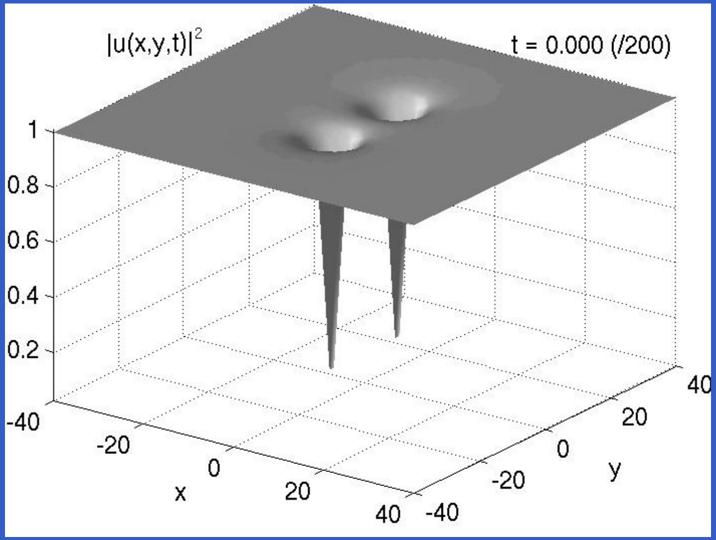


$\Rightarrow$



[movie]

- Same charge (vortex pair) $\rightarrow$ rotate about mid-point



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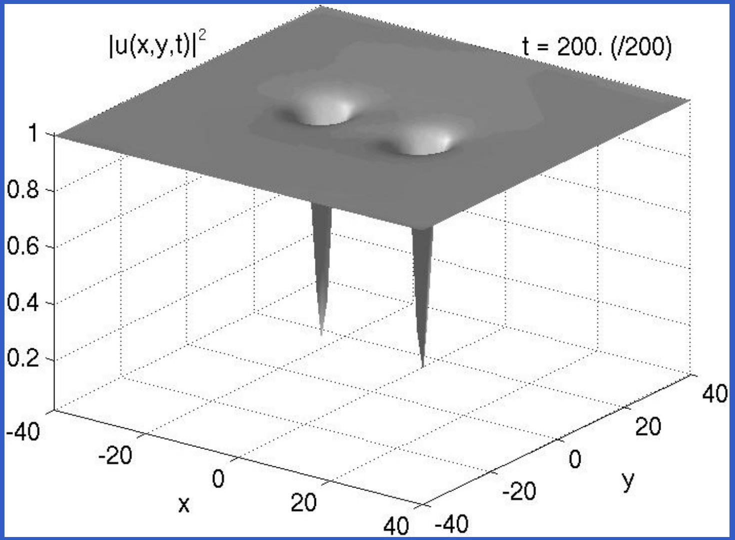

[movie]

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$$\dot{x}_1 = -\,\omega_{\rm vort}\, S_2 \frac{y_1 - y_2}{2r_{12}^2},$$

$$\dot{y}_1 = +\,\omega_{\rm vort}\, S_2 \frac{x_1 - x_2}{2r_{12}^2},$$

Vortices: vortex-vortex interactions

- Movement induced by phase gradient and density gradient [Matched asymptotics $\to$ Kivshar+Pismen+...].
- One vortex induces gradients on the other vortex $\to$ movement
- Gradient of vortex phase ($v_{\text{fluid}} \propto \nabla\phi$) is proportional to $1/\text{separation}$ and velocity is $\perp$ to line joining other vortex:

$$\dot{x}_1 = -\,\omega_{\text{vort}}\, S_2 \frac{y_1 - y_2}{2r_{12}^2},$$

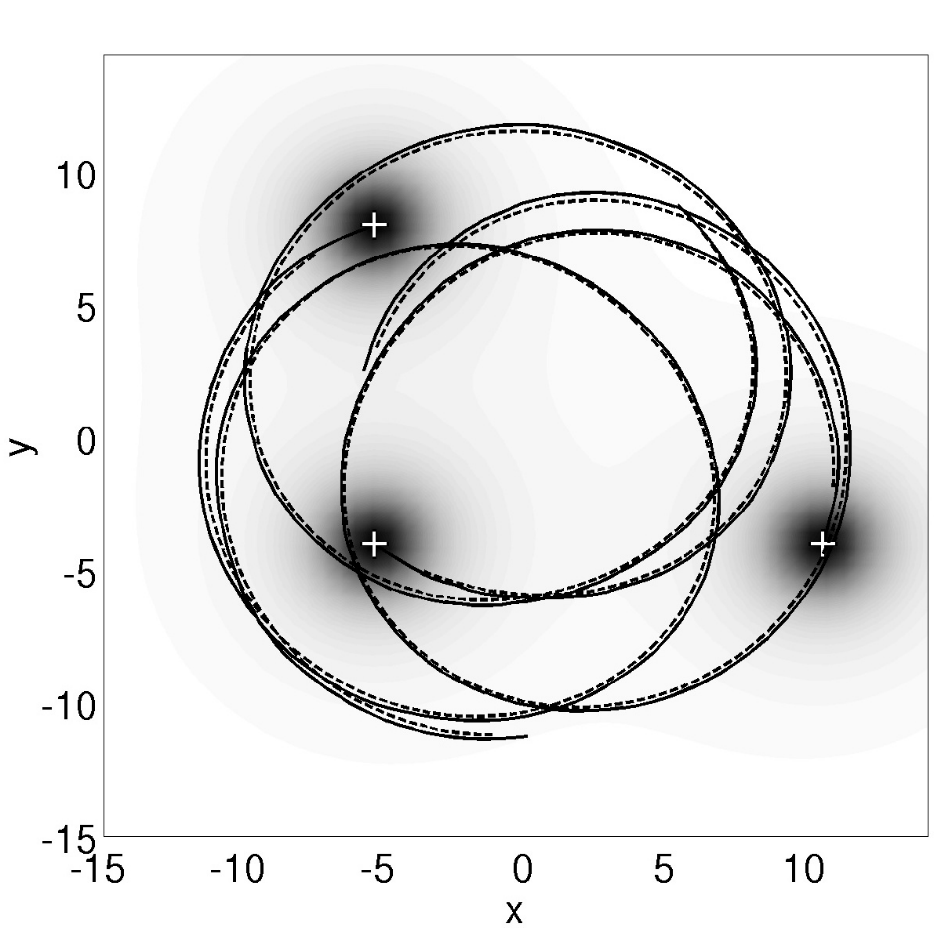
$$\dot{y}_1 = +\,\omega_{\text{vort}}\, S_2 \frac{x_1 - x_2}{2r_{12}^2},$$

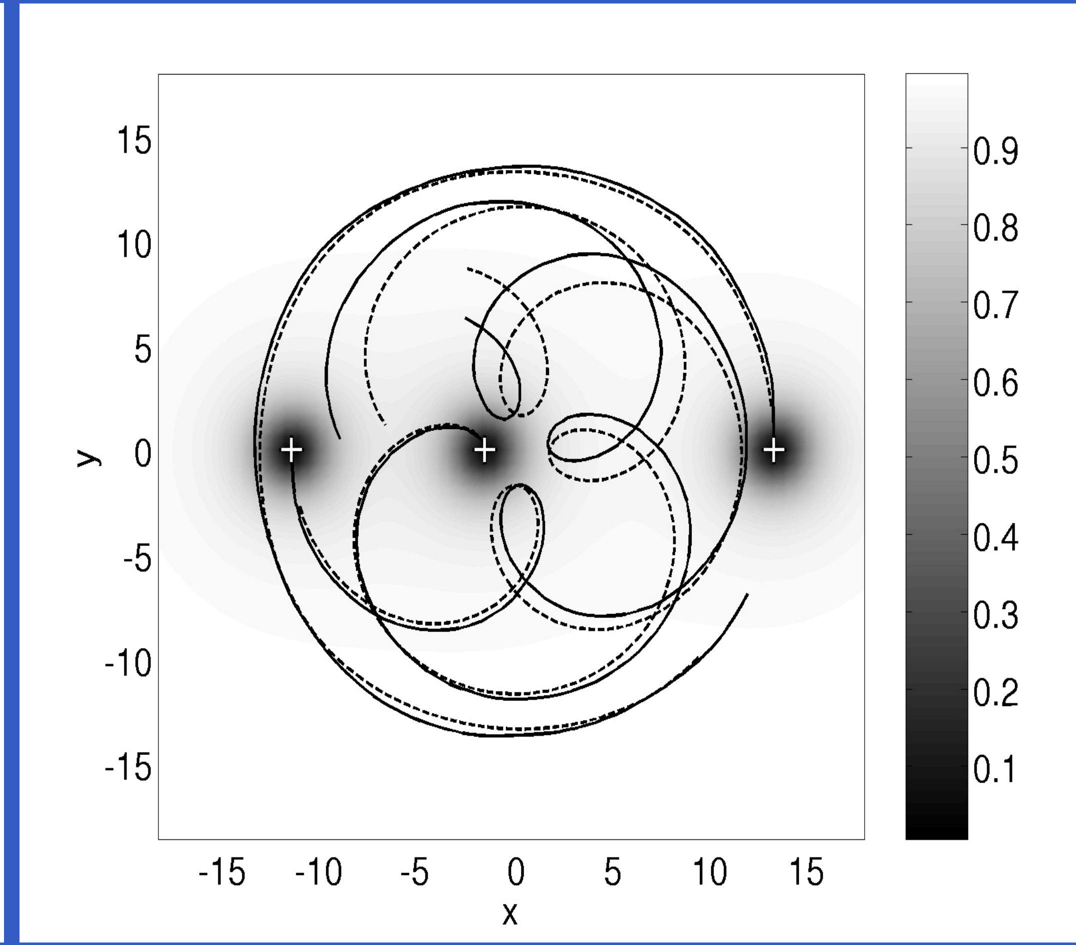
- Superposition of N vortices:

$$\dot{x}_m = -\frac{\omega_{\text{vort}}}{2} \sum_{n\neq m}^{N} S_m \frac{y_m - y_n}{r_{mn}^2}$$

$$\dot{y}_m = +\frac{\omega_{\text{vort}}}{2} \sum_{n\neq m}^{N} S_m \frac{x_m - x_n}{r_{mn}^2}$$

Vortex trajectories (no MT): PDE vs ODE





[movie] [movie]

Vortices in MT: reduced ODE dynamics

- Let us add all contributions:
  - vortex precession inside MT + vortex-vortex interactions

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\end{aligned}
$$

- Conserved quantities: Hamiltonian and angular momentum:

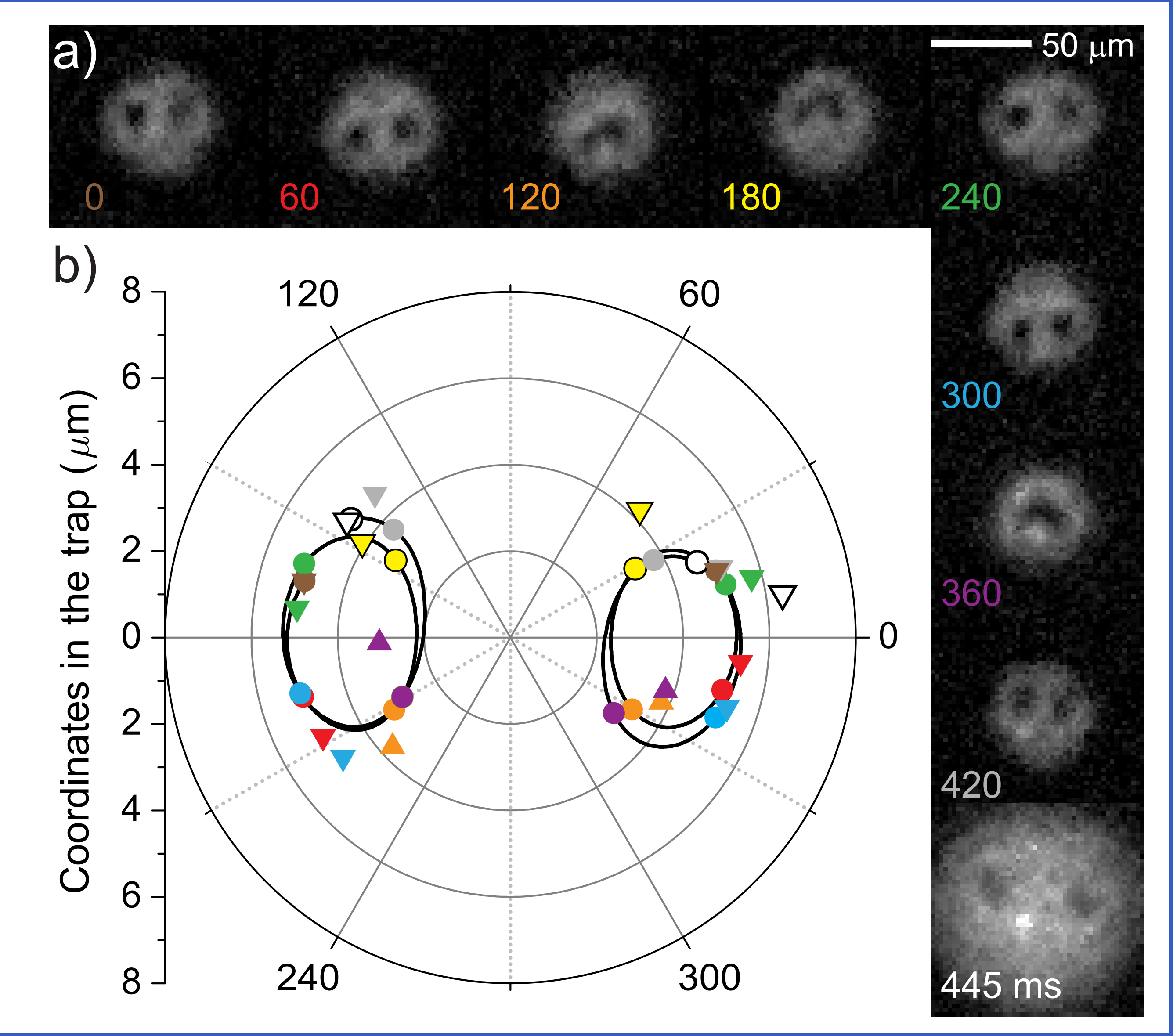
$$
\begin{aligned}
H &= -\frac{\omega_{\rm pr}^0}{2} \sum_{n=1}^{N} \ln(1-r_n^2) + \frac{\omega_{\rm vort}}{4} \sum_{n=1}^{N} \sum_{m\neq n}^{N} S_m S_n \ln(r_{mn}^2), \\
L_0 &= \sum_{n=1}^{N} S_n r_n^2.
\end{aligned}
$$

Vortex dipoles inside MT:

OPPOSITE charge pair:

$S_1 = 1$ & $S_2 = -1$

Experiments (David Hall) vs. theory



Stationary (non-rotating) equilibria:

- Equilibrium for diametrically opposed (symmetric) vortices:

$$r_{\rm eq} = 2\sqrt{\frac{\omega_{\rm vort}}{4\,\omega_{\rm pr}^0 + \omega_{\rm vort}}}$$

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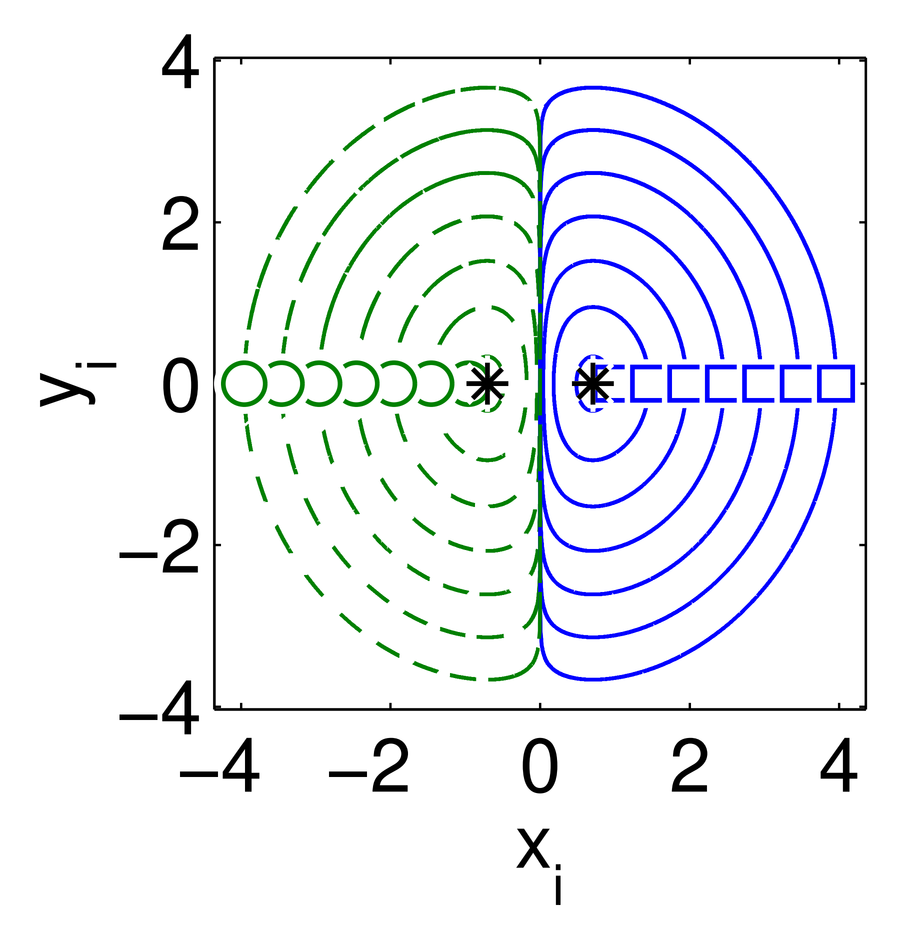
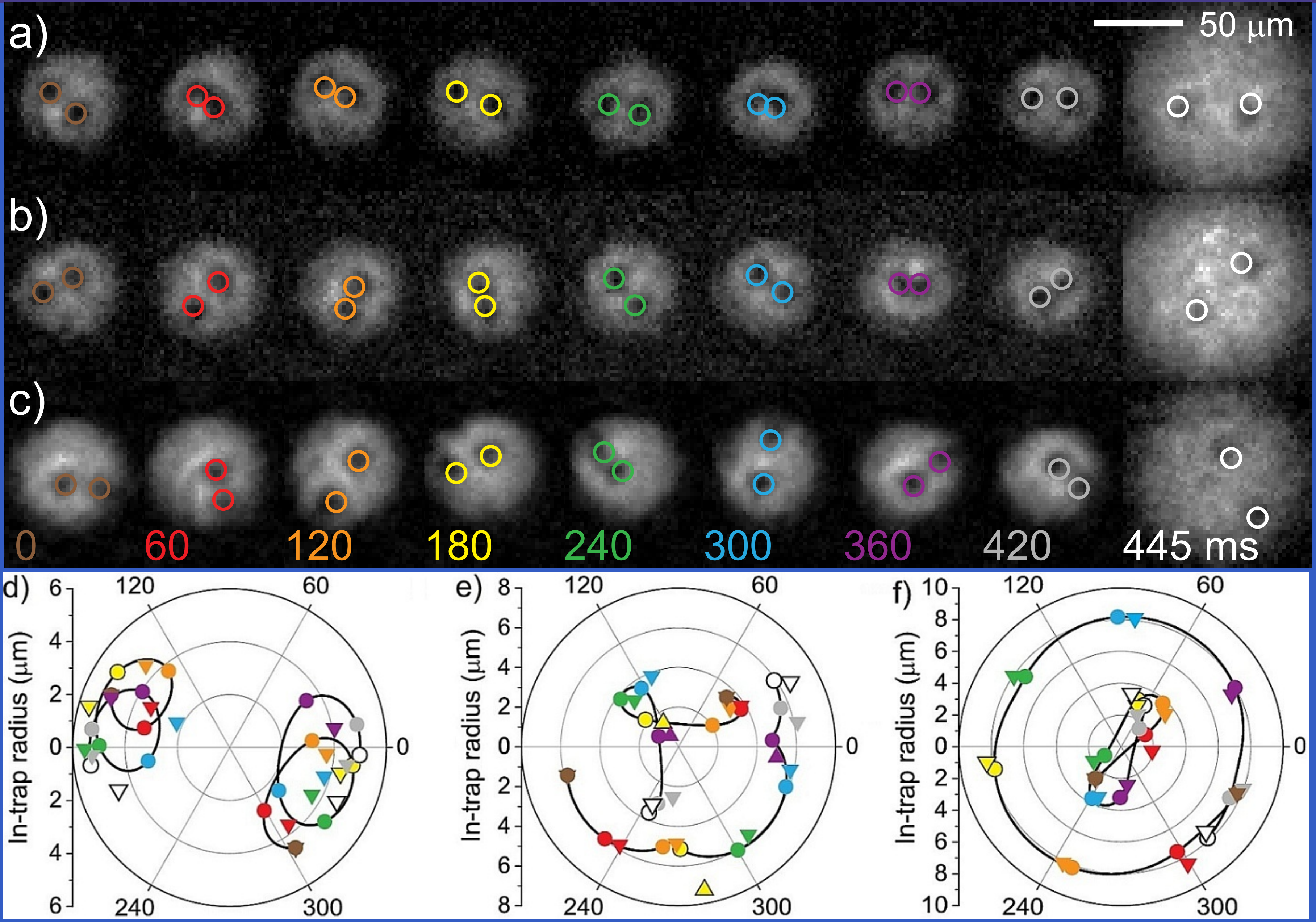


- Rotation frequecy around the equilibrium $\omega_{\rm eq} > 0 \Rightarrow$ always STABLE ! [contrast with the same-charge vortex case]

More experiments (David Hall) → the Spirograph



Asymmetric co-rotating equilibria:

- Consider diametrically opposed vortices but asymmetric wrt to the center: $z_1 = r_1 \exp(i\omega_{\rm orb} t)$ and $z_2 = -r_2 \exp(i\omega_{\rm orb} t)$ with $r_1 > r_2$.
- Asymmetric co-rotating equilibrium positions:

$$\omega_{\rm pr}^0 \left(\frac{1}{1 - r_1^2} + \frac{1}{1 - r_2^2} \right) - \frac{\omega_{\rm vort}}{2 r_1 r_2} = 0.$$

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- Rotating with freq:

$$\omega_{\rm orb} = \frac{1}{2} \left[\omega_{\rm pr}^0 (\alpha - \beta) + \gamma \omega_{\rm vort} \left(\frac{r_1}{r_2} - \frac{r_2}{r_1} \right) \right],$$

where

$$\alpha = \frac{1}{1-r_1^2}, \quad \beta = \frac{1}{1-r_2^2}, \quad \text{and} \quad \gamma = \frac{1}{2r_{12}^2}.$$

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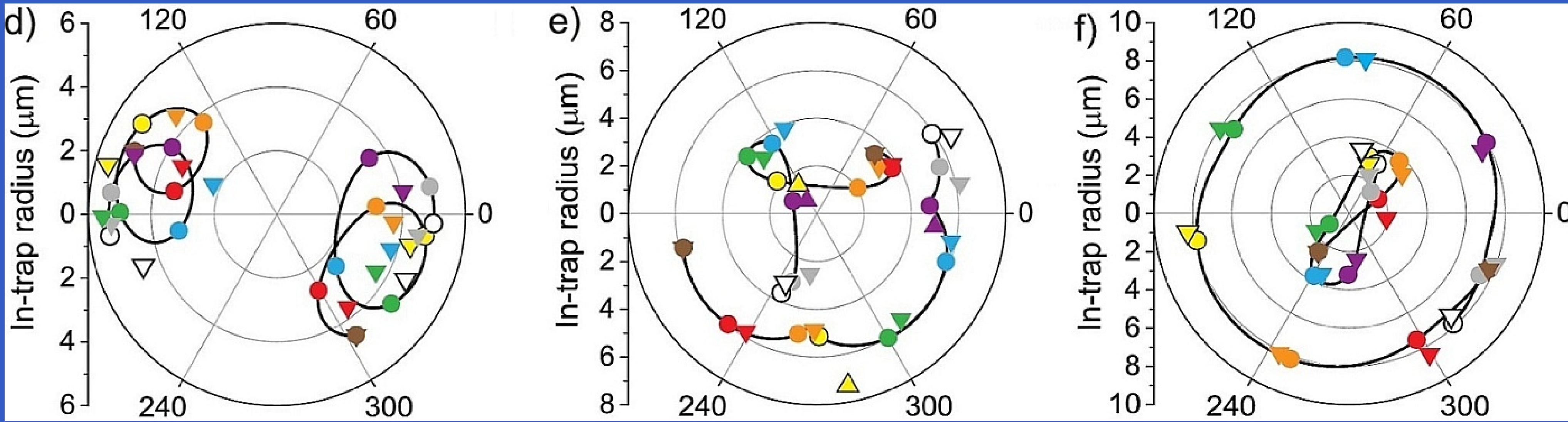
- In co-rot frame: perturbs about equilibria result in rotating orbits.
$\rightarrow$ epitrochoidal motion (Spirograph) in the original frame !!!
$\rightarrow$ Generic motion: quasi-periodic epitrochoids.

Motion about asymmetric equilibria: epitrochoids!

- On the original (lab.) frame. Experiments:



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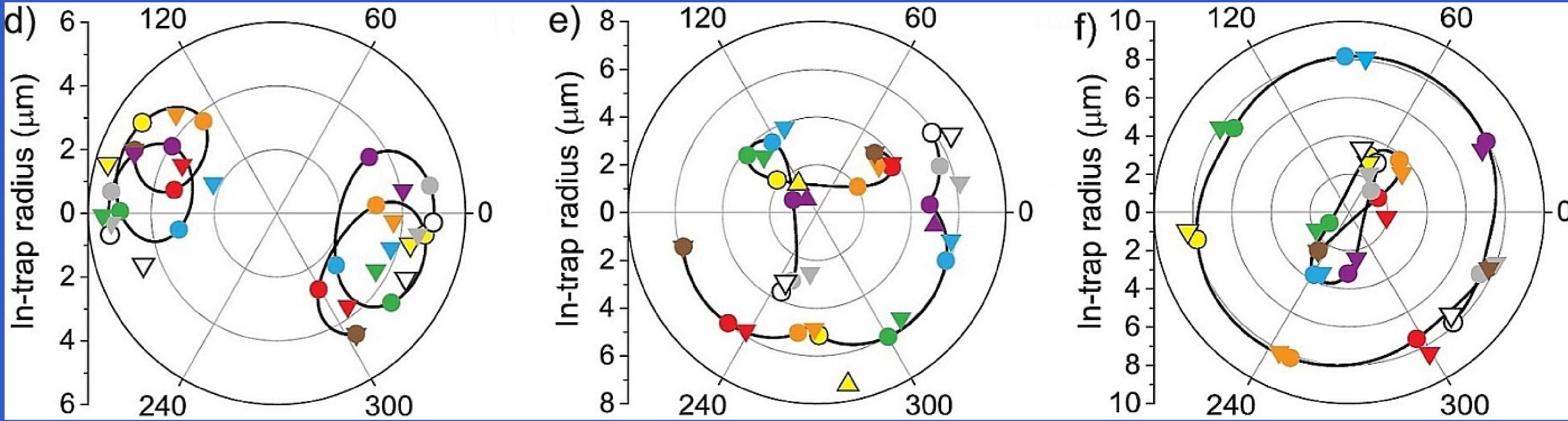
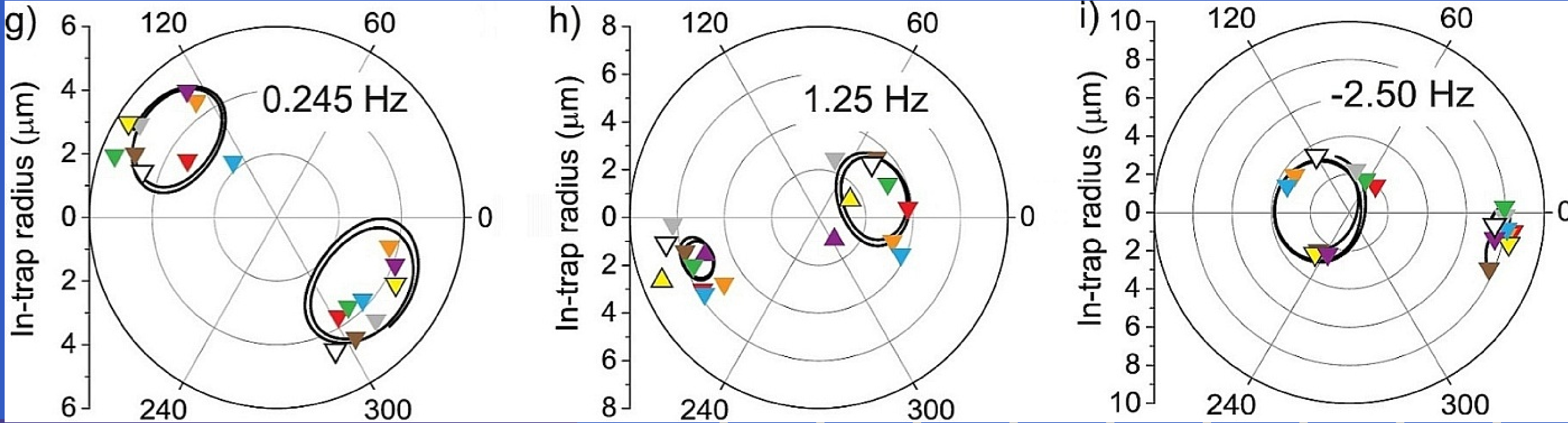


- On the co-rotating ($\omega_{\rm orb}$) reference frame:



Vortex pairs inside MT:

SAME charge pair:

$$S_1 = 1 = S_2$$

Adim. and transform to co-rotating polar coord.:

- Adimensionalize: $(X,Y)=\dfrac{(x,y)}{R_{\rm TF}}, \quad \tau=\omega_{\rm pr}^0 t, \quad c\equiv\dfrac{1}{2}\dfrac{\omega_{\rm vort}}{\omega_{\rm pr}^0}.$
- Transf. co-rot to polar: $X_n = r_n\cos\theta_n,\ Y_n = r_n\sin\theta_n,\ \delta_{mn}=\theta_m-\theta_n$

$$\dot r_m = -\frac{c\, r_n \sin\delta_{mn}}{r_{mn}},$$

$$\dot\delta_{mn} = \left(r_m^2 - r_n^2\right)\frac{c\cos\delta_{mn}}{r_m\, r_n\, r_{mn}^2} + \frac{1}{1-r_m^2} - \frac{1}{1-r_n^2}.$$

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- Steady state is $r_1=r_2=r_*$ (for ANY r_*) and $\theta_1-\theta_2=\pi$, in co-rot:

$$\omega_{\rm orb}\equiv\dot\theta_1=\dot\theta_2=\frac{c}{2r_*^2}+\frac{1}{1-r_*^2}.$$

[$\omega_{\rm orb}>0 \Rightarrow \nexists$ no-rotating equilibria $\rightarrow$ only ROTATING equilibria]

Epitrochoids about SYMMETRIC rotating equilibria:

- Perturbations about equilibria: $r_m = r_* + R_m$ and $\delta_{mn} = \pi + \Delta_m$

- Eqs on the pert.:

$$
\begin{aligned}
\ddot{R}_m &= -\omega_{\rm ep}^2 \left(R_n - R_m\right), \\
\ddot{\Delta}_m &= -\omega_{\rm ep}^2 \left(\Delta_m - \Delta_n\right), \\
\omega_{\rm ep}^2 &= \frac{c^2}{2r_*^4} - \frac{2c}{\left(1 - r_*^2\right)^2},
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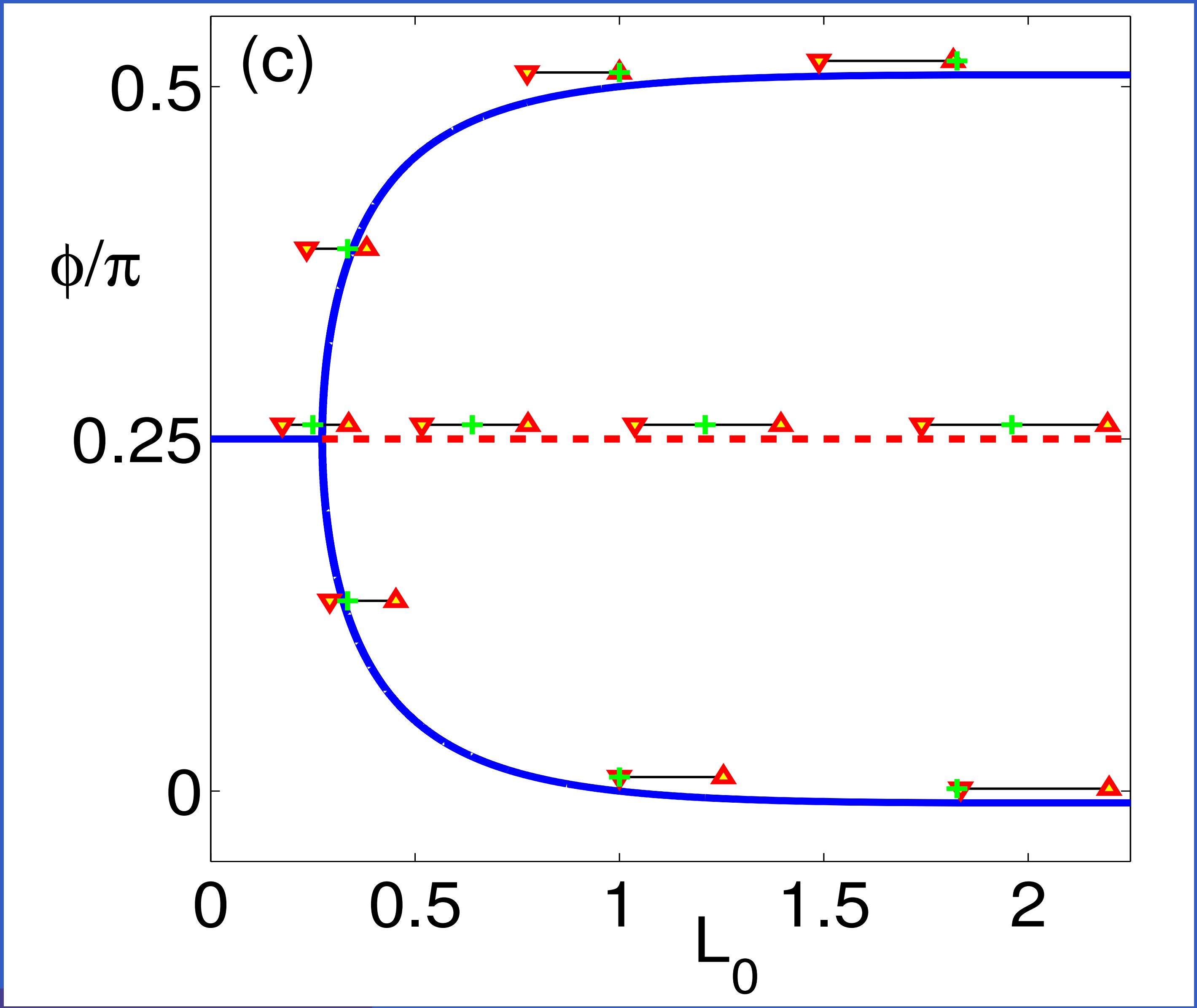
- If $r_* > r_{\rm crit} \Rightarrow$ INSTABILITY !!!

  $\rightarrow$ What does it imply for dynamics ?

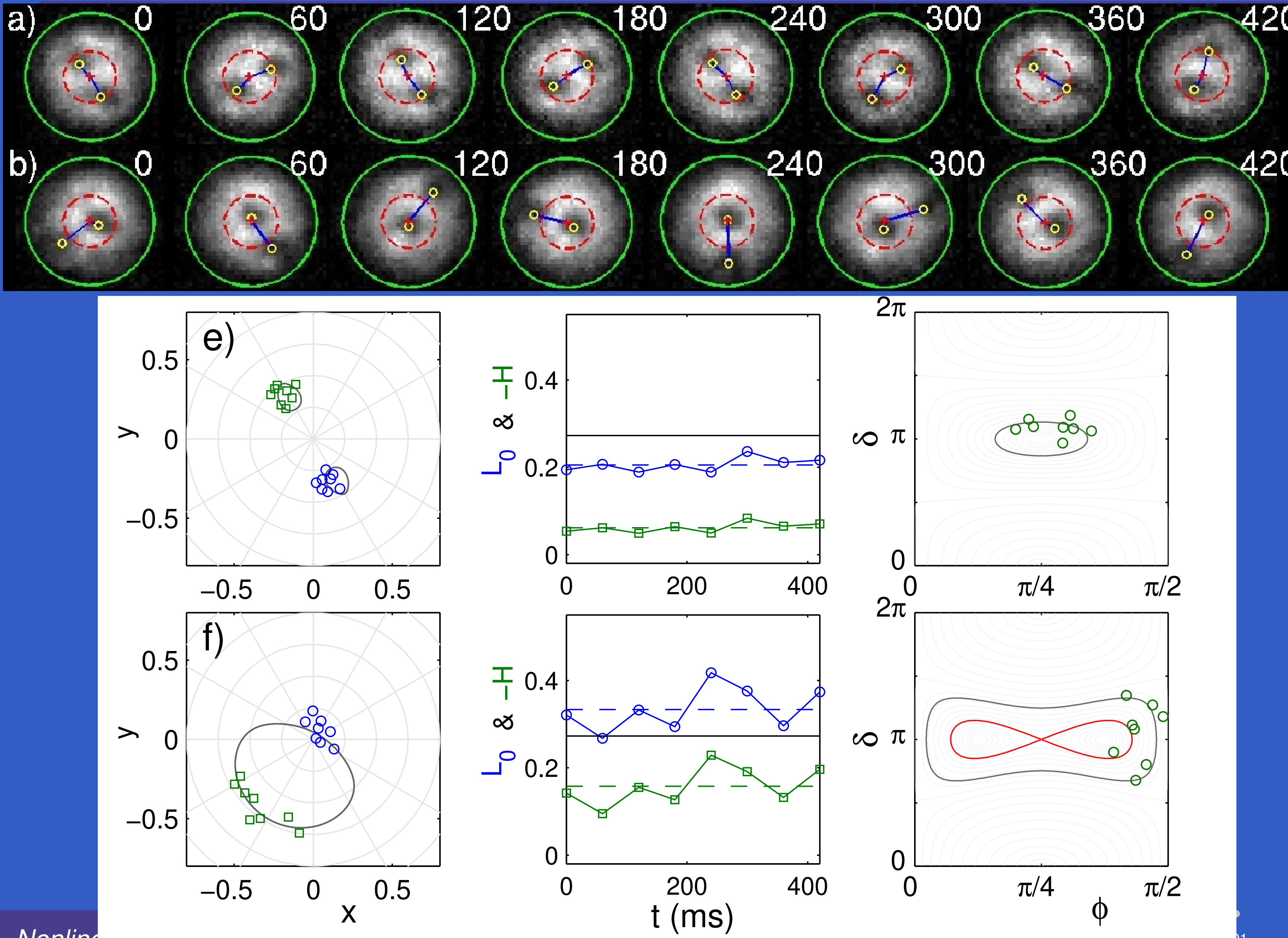
  $\rightarrow$ Q: Can this be observed in experiment ???

Bif. of equilibria vs. angular momentum: 2 vortices

- Angular momentum $L_0 = r_1^2 + r_2^2$ and $\tan\phi = r_2/r_1$ (polar coord.)

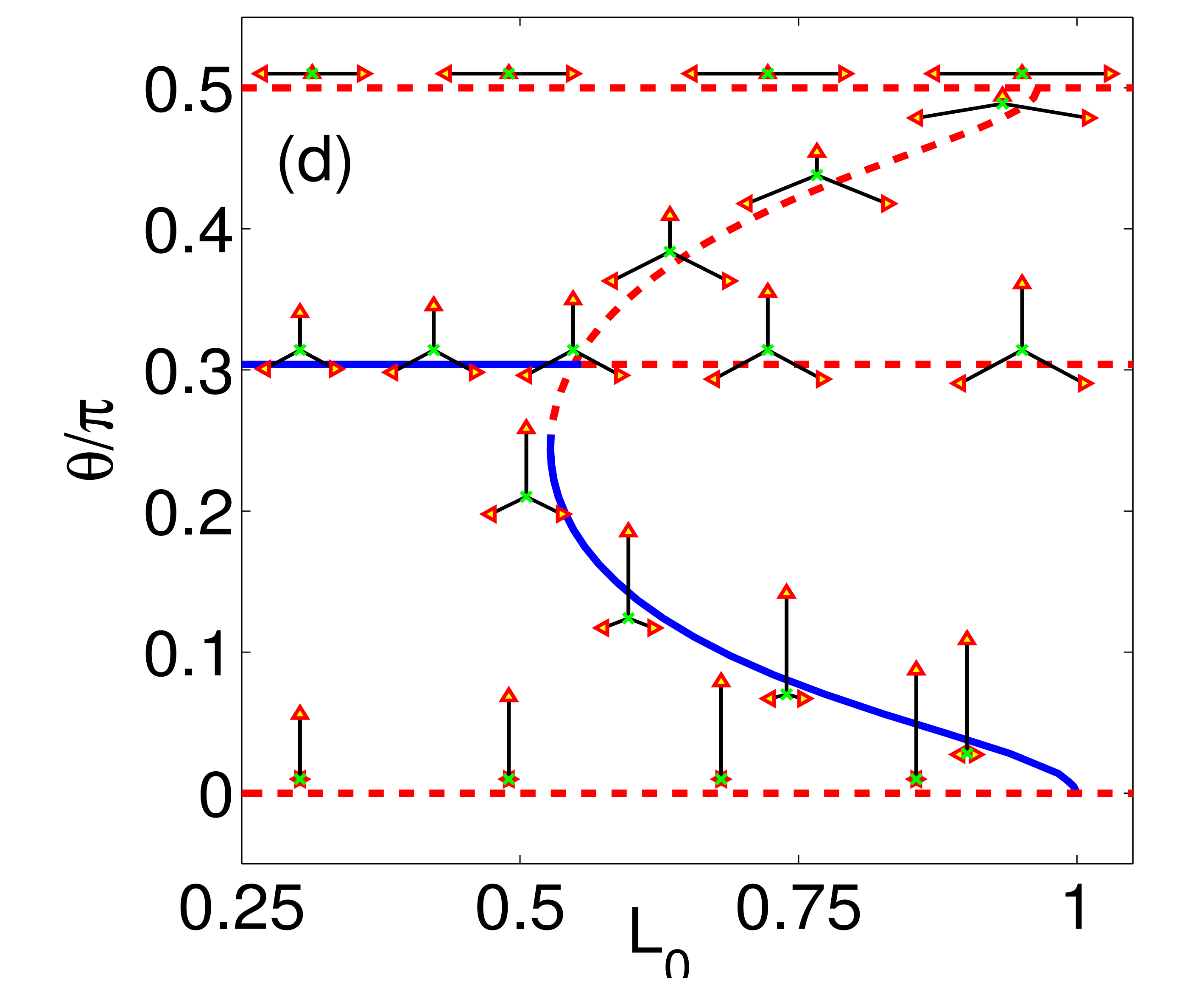


EXPERIMENTAL results: hunting for the pitchfork bifurcation (for $N = 2$)


a)
0
60
120
180
240
300
360
420
b)
0
60
120
180
240
300
360
420
e)
f)
0.5
0
−0.5
y
x
L_0 & −H
0.4
0.2
0
200
400
t (ms)
2π
π
δ
0
π/4
π/2
ϕ

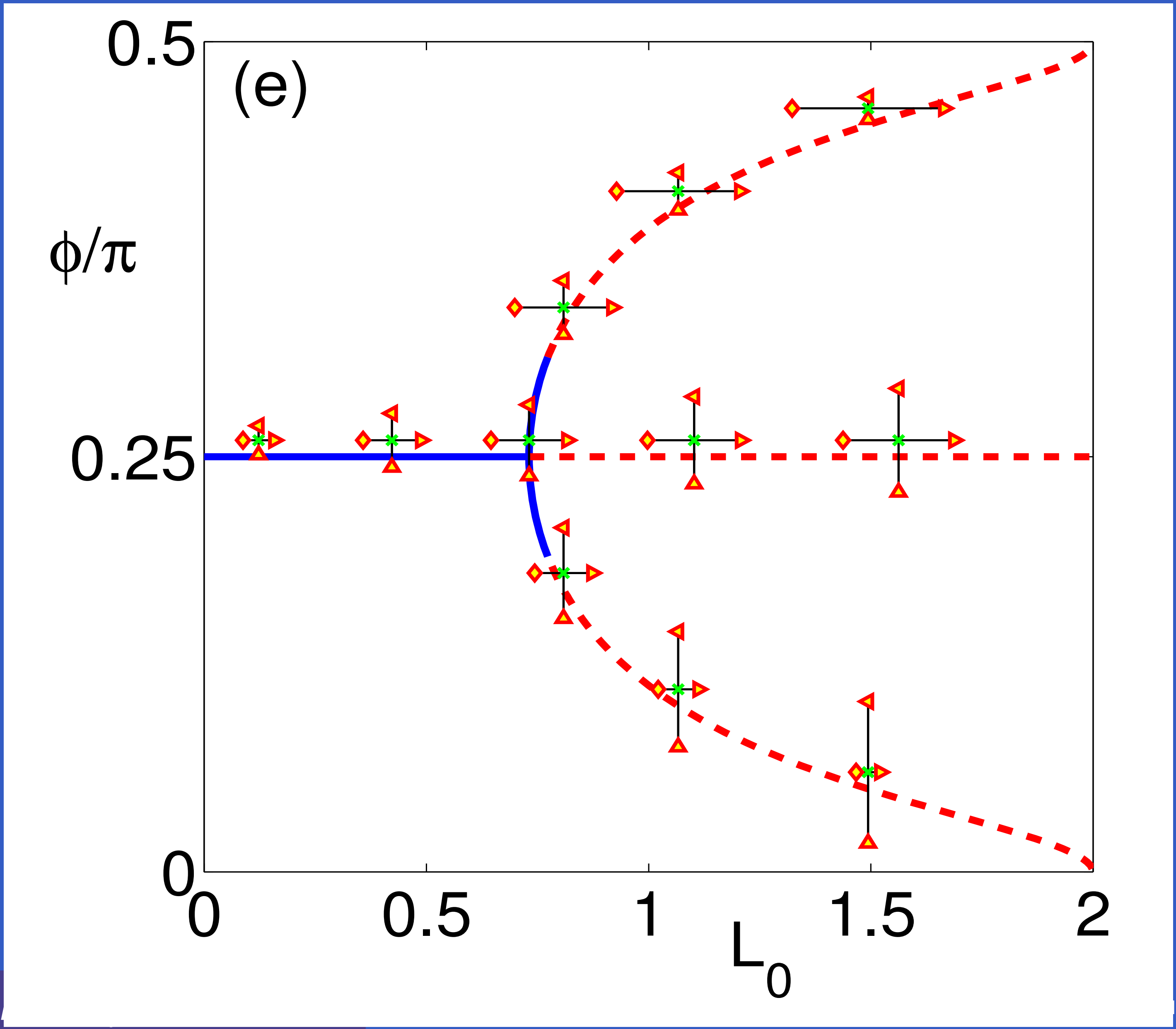
Bif. of equilibria vs. angular momentum: 3 vortices

- Angular momentum $L_0 = r_1^2 + r_2^2 + r_3^2$ and $\tan\phi = r_2/r_1$ (polar coord.)

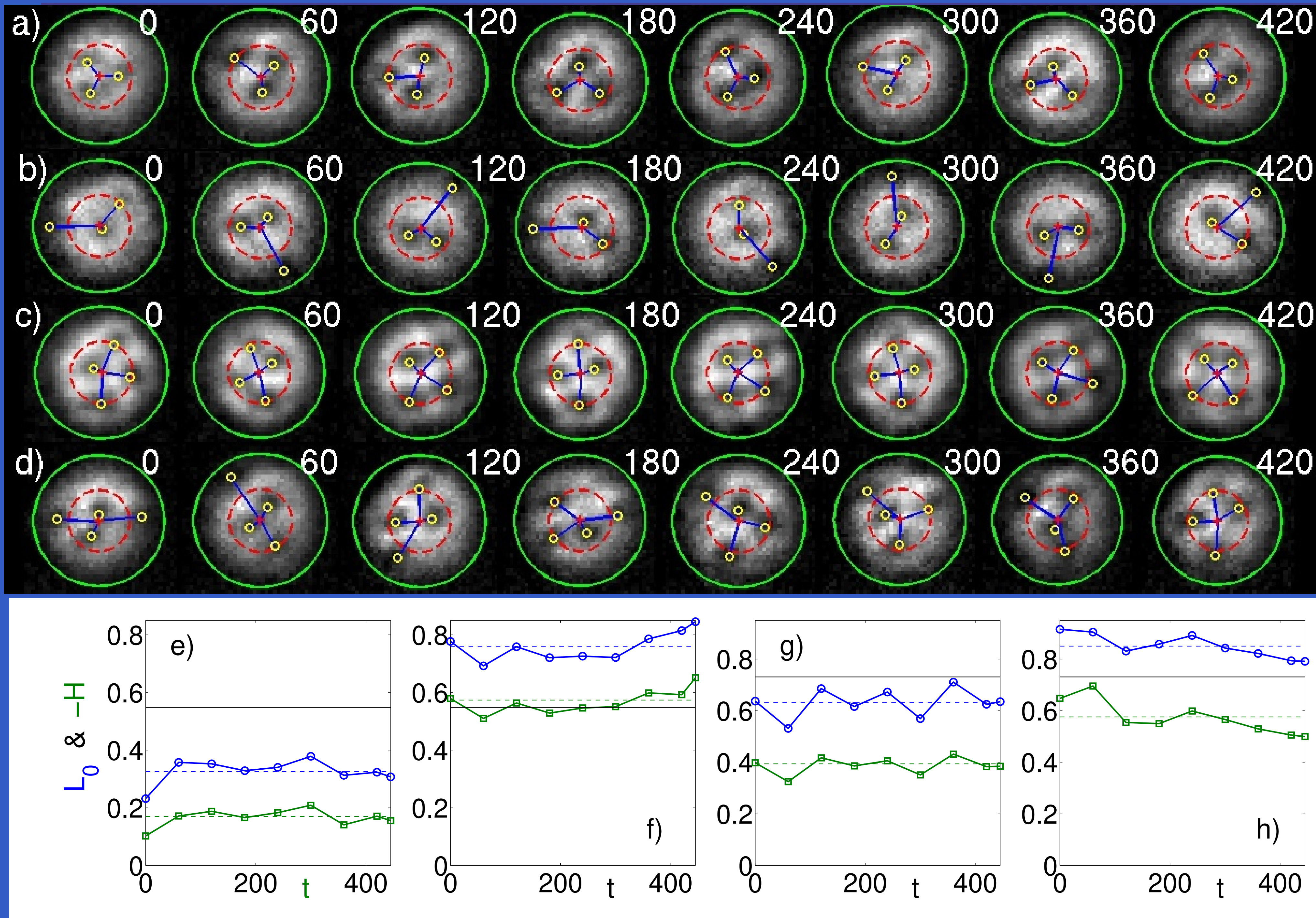


Bif. of equilibria vs. angular momentum: 4 vortices

- Angular momentum $L_0 = r_1^2 + r_2^2 + r_3^2 + r_4^2$



Back to EXPERIMENTS: hunting for the pitchfork bifurcation (for $N > 2$)


a) 0 60 120 180 240 300 360 420
b) 0 60 120 180 240 300 360 420
c) 0 60 120 180 240 300 360 420
d) 0 60 120 180 240 300 360 420
e)
f)
g)
h)
L_0 & $-H$
t
0 0.2 0.4 0.6 0.8
0 200 400

EXTENSIONS:

loss & gain

---

anisotropy

---

non-Euclidean geometries

Dissipation ($T \neq 0$) $\Rightarrow$ Vortex outward spiralling.

- Phenomenological dissipation ($\gamma > 0$):

$$(i-\gamma)u_t = -\frac{1}{2}\nabla^2 u + V(r)u + |u|^2 u.$$

- Energy loss: $\frac{dE}{dt} = -2\gamma \iint |u_t|^2 dx\, dy$

$$\cdots \Rightarrow \begin{cases} \dot{x} = -S\omega_{\rm pr}\, y + \tilde{\gamma}\, \dot{y}, \\ \dot{y} = S\omega_{\rm pr}\, x - \tilde{\gamma}\, \dot{x}, \end{cases}$$

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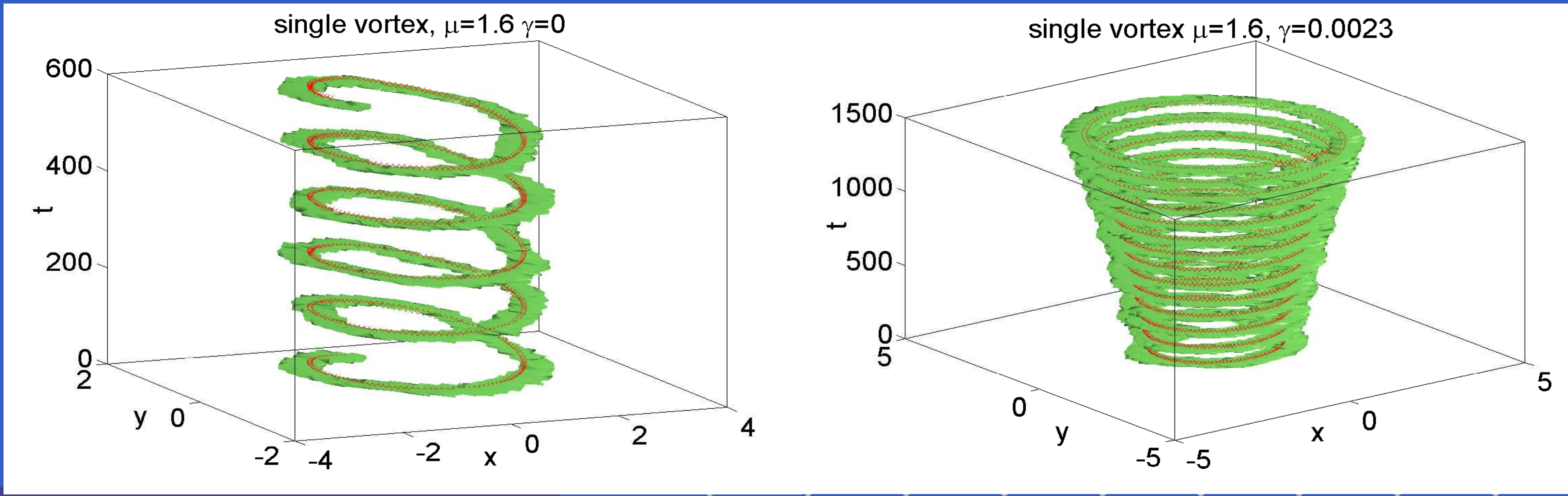
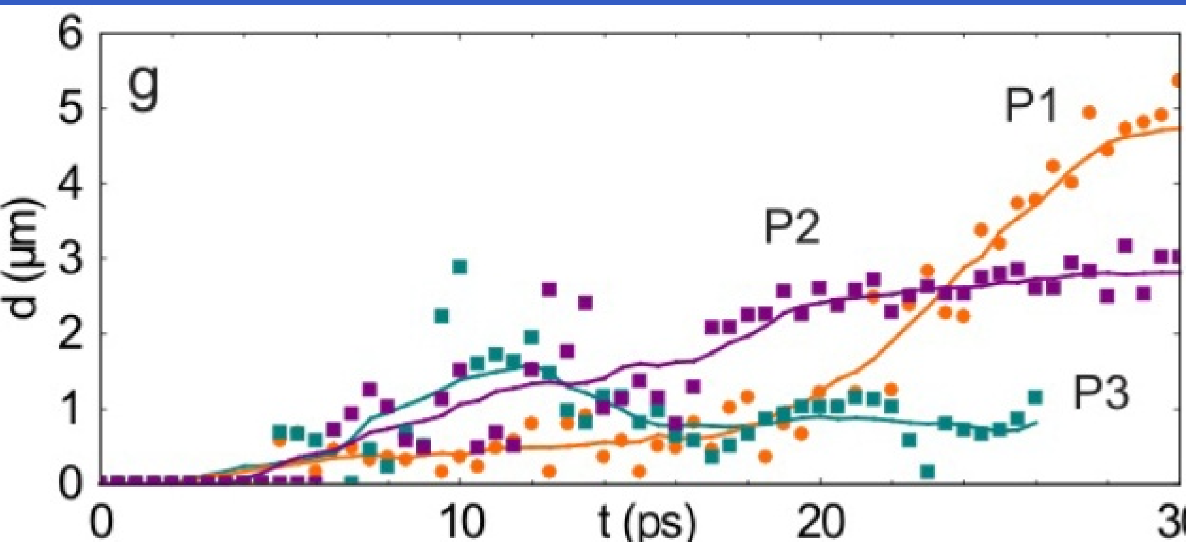


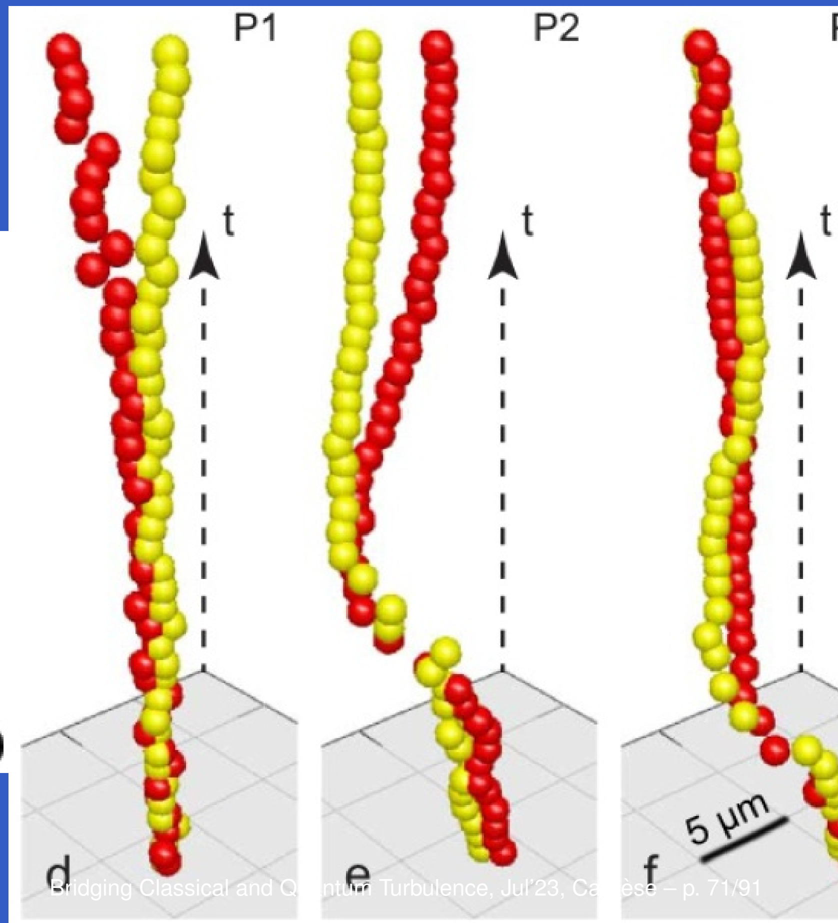
Polariton BECs ⇒ Loss + Gain → spiralling [Sanvitto].

- Experimental results in (spinor) exciton(ϕ)-polariton(ψ) condensates

$$i\frac{\partial \phi_\pm}{\partial \tau} = \left(-\epsilon\frac{1}{2}\nabla^2 - i\gamma_\psi + |\phi_\pm|^2 + \bar{\alpha}|\phi_\mp|^2\right)\phi_\pm + \omega_R\,\psi_\pm$$

$$i\frac{\partial \psi_\pm}{\partial \tau} = \left(-\frac{1}{2}\nabla^2 - i\gamma_\phi\right)\psi_\pm + \omega_R\,\phi_\pm + \beta\left(\frac{\partial}{\partial X} \pm i\frac{\partial}{\partial Y}\right)^2 \psi_\mp + F_\pm,$$





Anisotropy $\rightarrow$ complex periodic orbits

N vortices with positions (x_i, y_i) in an anisotropic trap

$$
\begin{aligned}
\dot{x}_k &= -s_k Q\omega_y^2 y_k + \frac{B}{2}\sum_{j\neq k} s_j \frac{y_j - y_k}{\rho_{jk}^2}, \\
\dot{y}_k &= s_k Q\omega_x^2 x_k - \frac{B}{2}\sum_{j\neq k} s_j \frac{x_j - x_k}{\rho_{jk}^2},
\end{aligned}
$$

where $\rho_{jk}^2 = (x_j - x_k)^2 + (y_j - y_k)^2$ and (after rescaling) $Q\omega_x^2 = 1$ and $Q\omega_y^2 = 1 + \epsilon$.

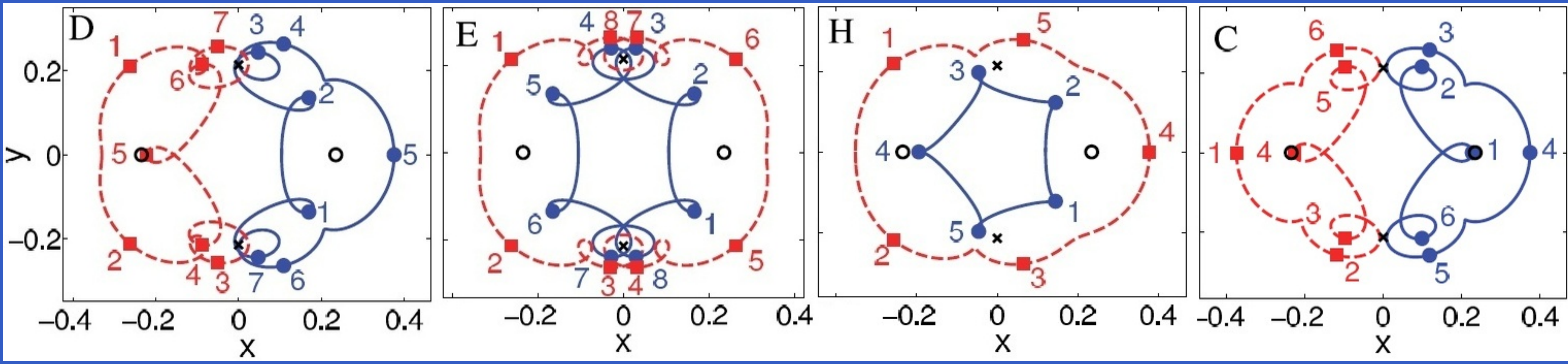
TWO opposite charge vortices (dipole) $\rightarrow$ fixed points are:

$$
\vec{P}_x = \pm\sqrt{\frac{B}{4(1+\epsilon)}}(0,0,1,-1) \quad \text{and} \quad \vec{P}_y = \pm\frac{\sqrt{B}}{2}(1,-1,0,0).
$$

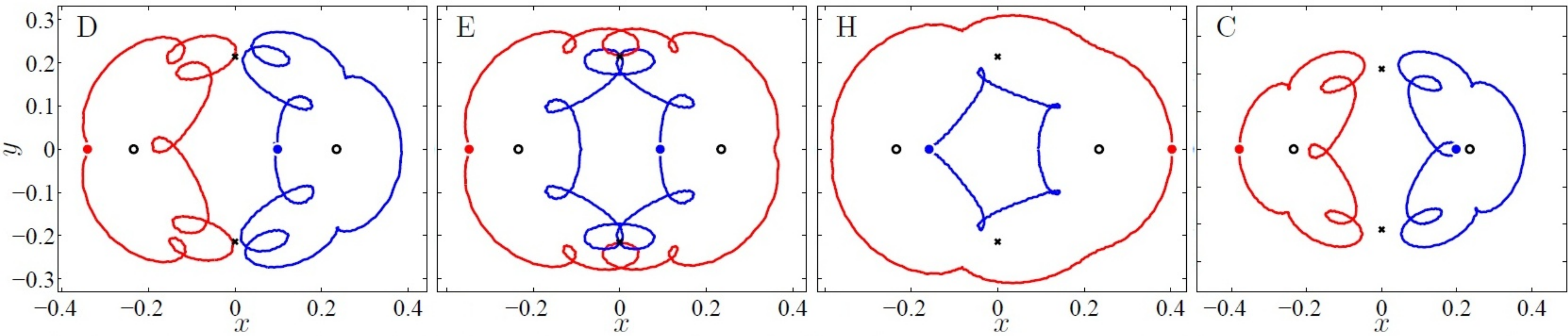
Reducing the system to a pendulum coupled to harmonic oscillator
$\Rightarrow$ conserved quantities $\Rightarrow$ approx Poincaré map for periodic orbits
$\Rightarrow$ determination of ICs leading to complicated periodic orbits.

Anisotropy → complex periodic orbits → ODE vs PDE

ODE:

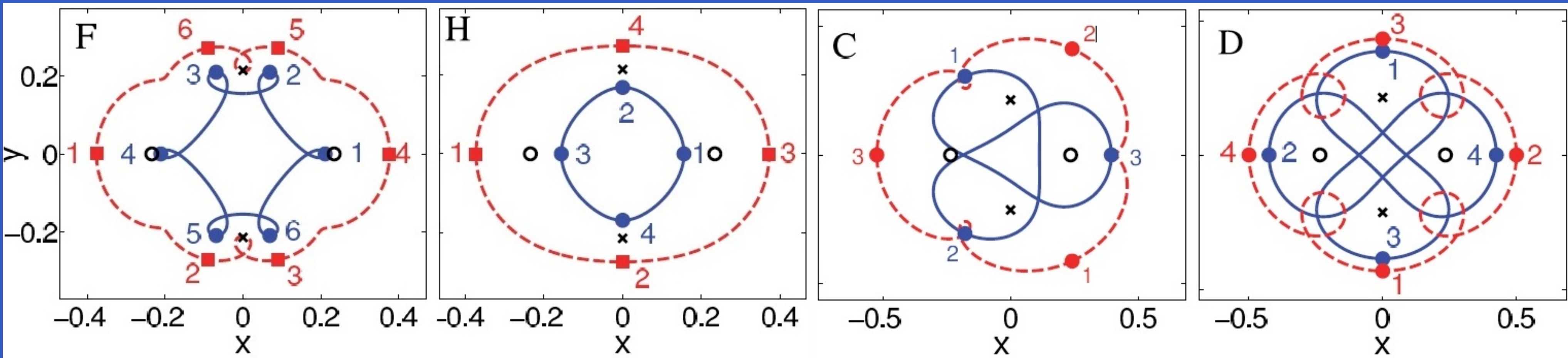


PDE:

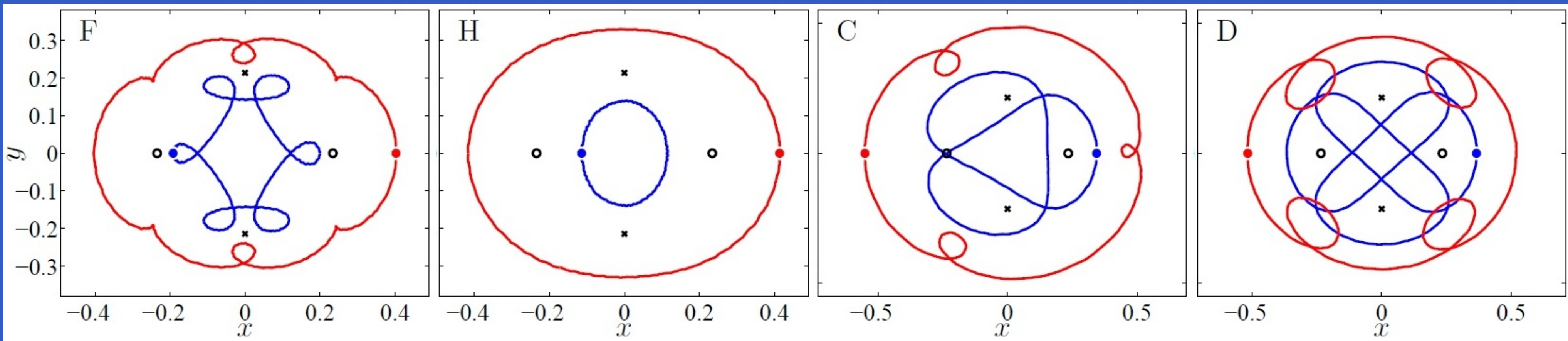


Anisotropy → complex periodic orbits → ODE vs PDE

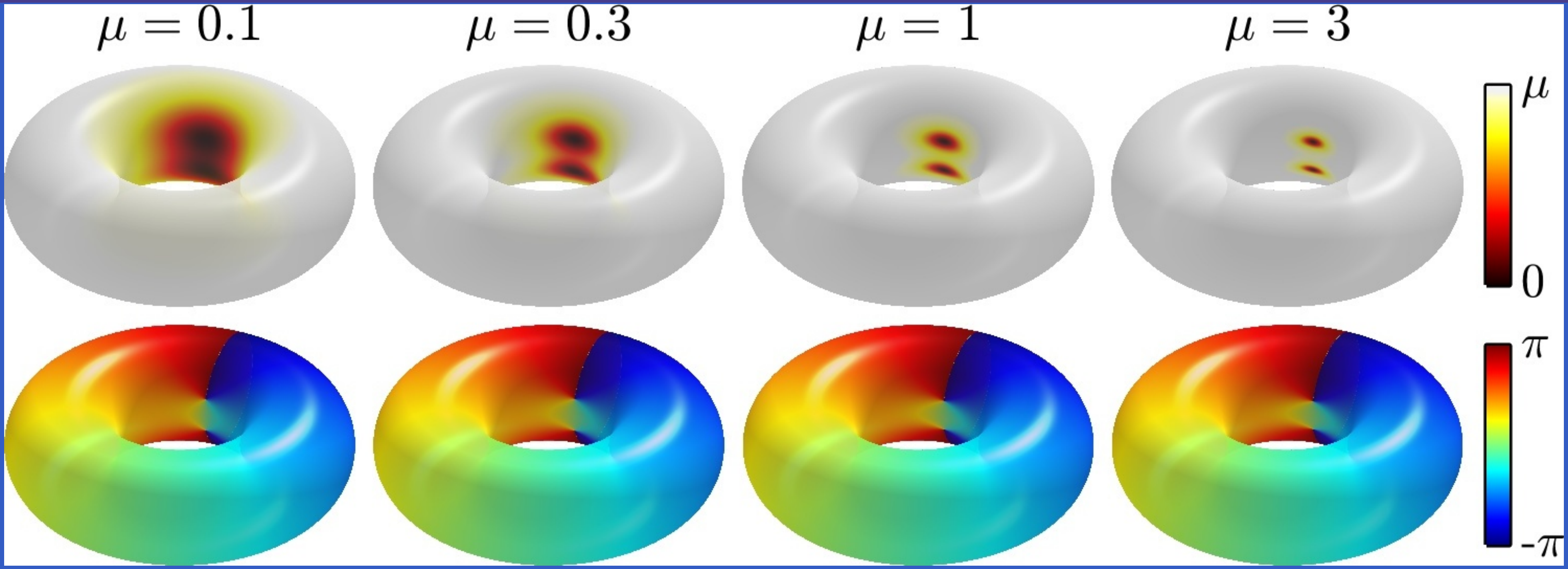
ODE:

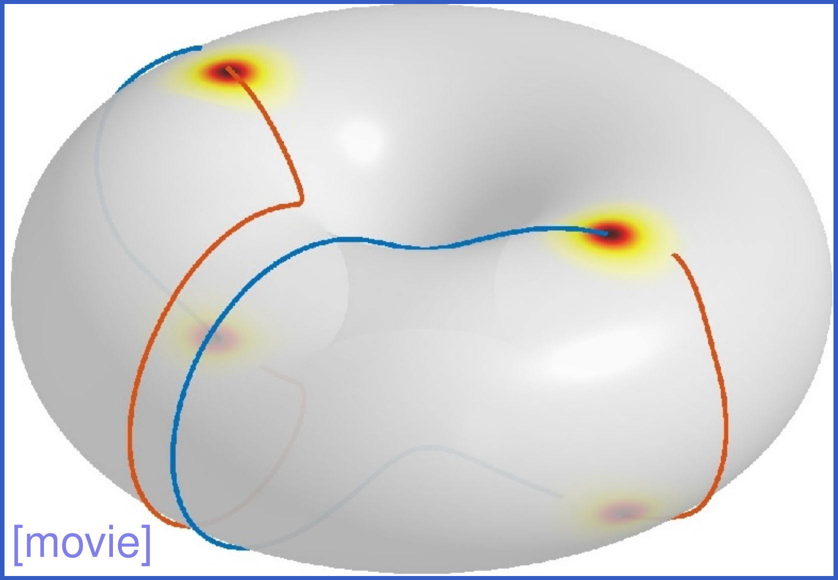


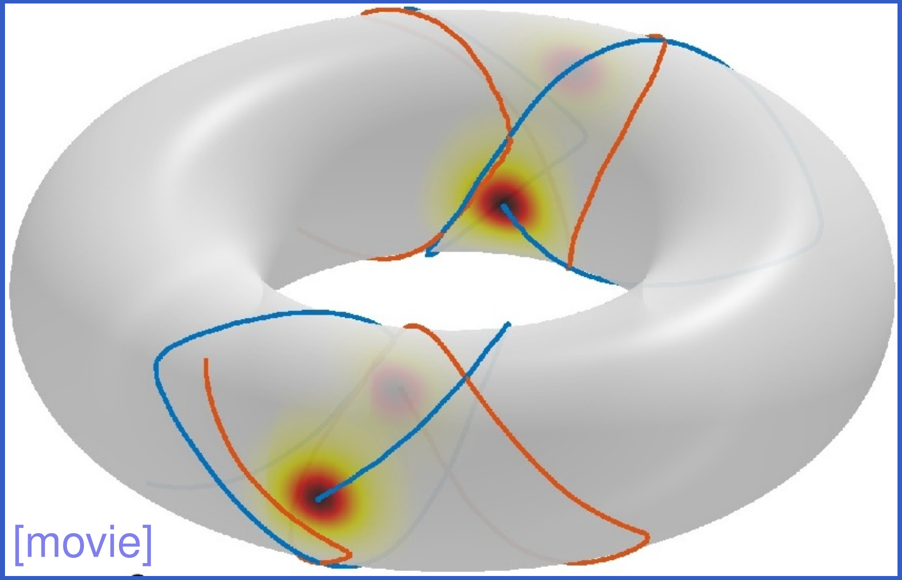
PDE:



Vortex dynamics in non-Euclidean geometries



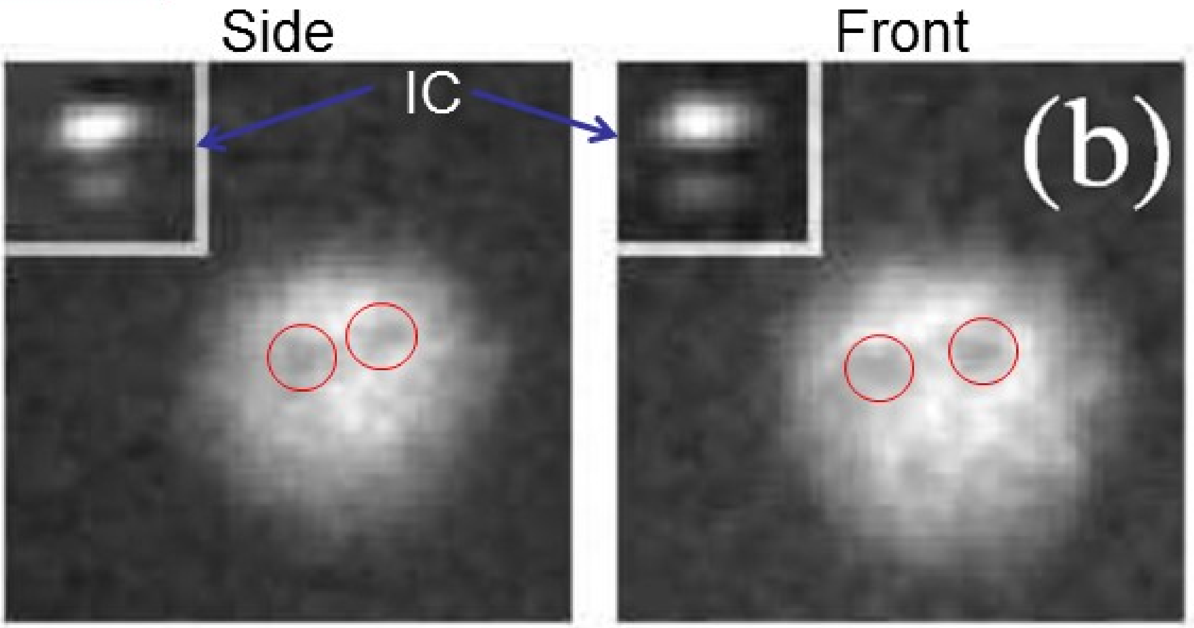




Vortex Rings in 3D BECs

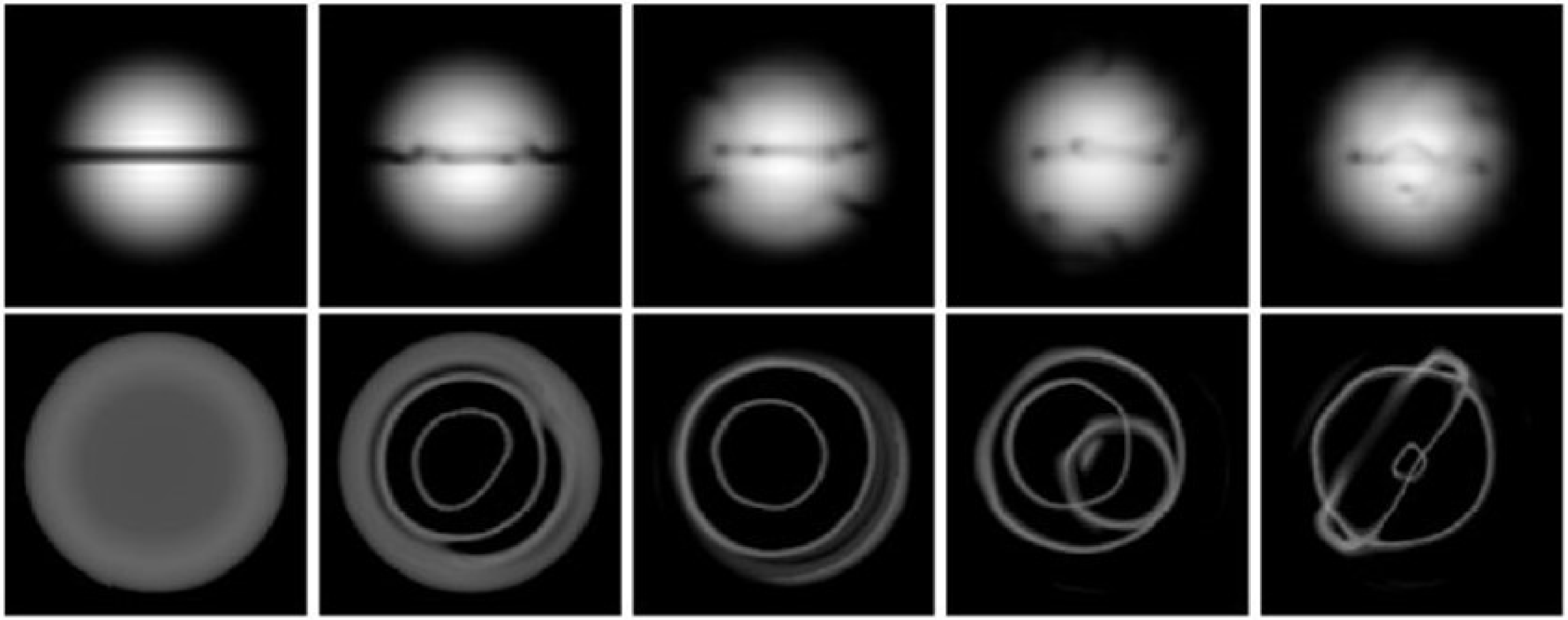
Physical Motivation

Vortex Rings BECs



• VRs observed in (many) experiments:

PRL **86** (2001) 2926. “Watching Dark Solitons Decay into VRs in a BEC”, B.P. Anderson et al.



Particle picture: NLS + Madelung = superfluid

- NLS + Madelung transformation ($u = \sqrt{\rho}\, e^{i\phi}$)
- Def. $v = \nabla\phi \rightarrow$ fluid velocity
- NLS $\rightarrow$ Non-viscous Eulerian fluid + “quantum pressure”
- Quant. press. $\propto \nabla^2\sqrt{\rho}/\sqrt{\rho} \rightarrow$ negligible away from vortex cores

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- NLS $\to$ Non-viscous Eulerian fluid + "quantum pressure"
- Quant. press. $\propto \nabla^2\sqrt{\rho}/\sqrt{\rho} \to$ negligible away from vortex cores
- Let us borrow technology from classical fluids: Biot-Savart law:

$$\mathbf{u}_v(\mathbf{x}) = \frac{1}{4\pi}\int \frac{\boldsymbol{\omega}(\mathbf{x}') \times (\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^3}\, d\mathbf{x}', \tag{3}$$

  where ω=vorticity and integral is computed over all of space.
- Consider a vorticity tube with circulation κ along a closed curve **R** parametrized by ℓ:

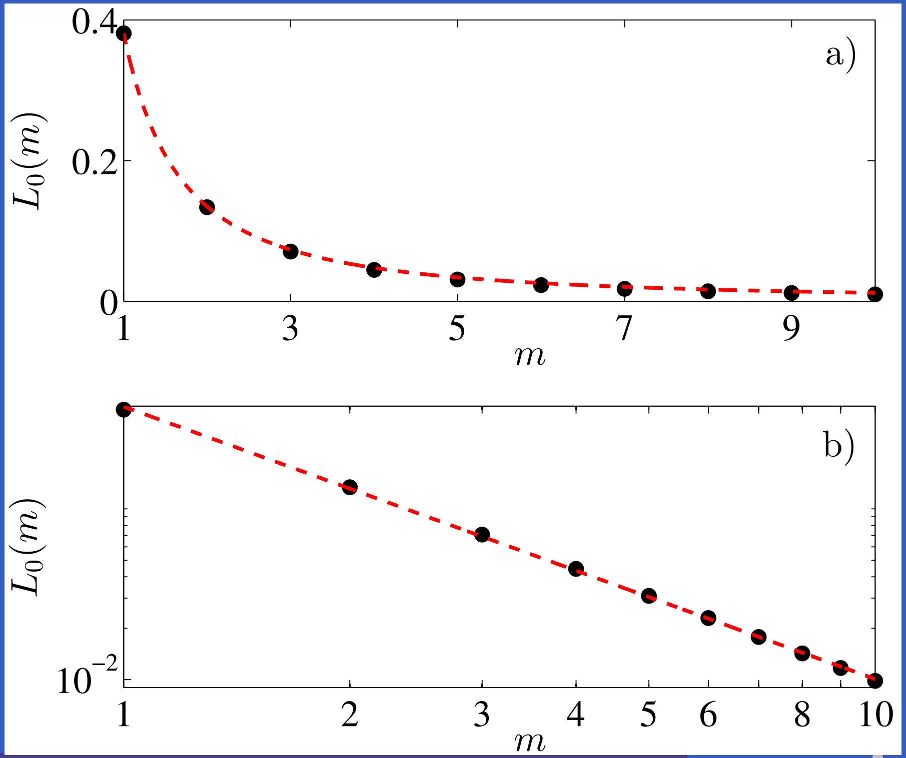
$$\mathbf{u}_v(\mathbf{x}) = \frac{\kappa}{4\pi}\oint \frac{\mathbf{s} \times (\mathbf{x} - \mathbf{R}(\ell))}{|\mathbf{x} - \mathbf{R}(\ell)|^3}\, d\ell \tag{4}$$

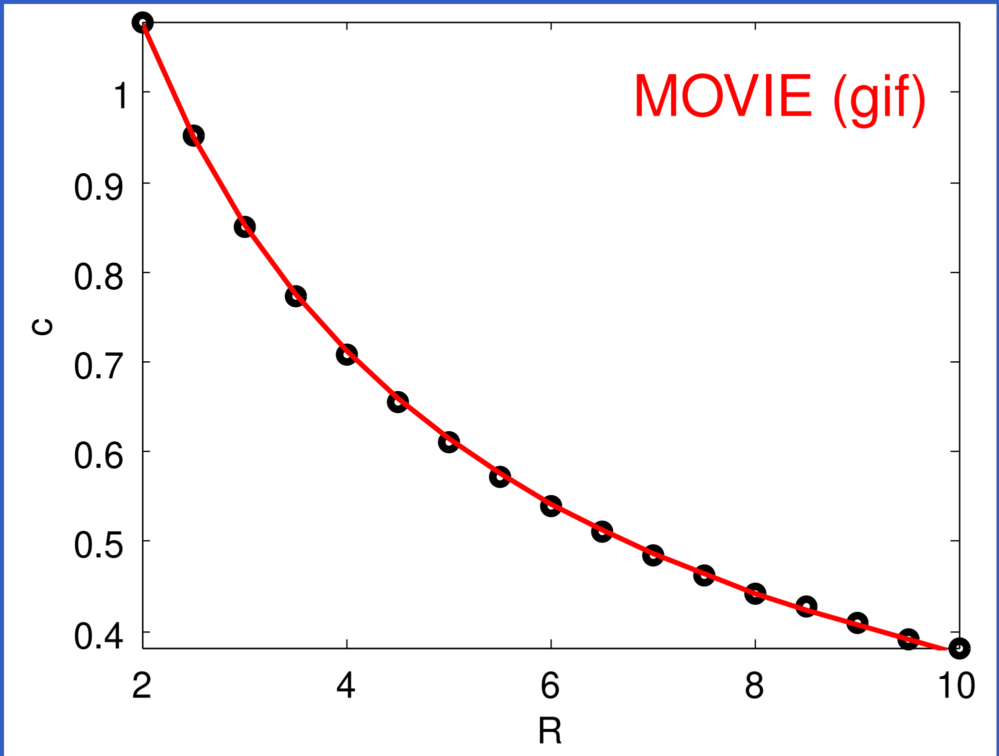
Single VR → self-induced velocity

- The Biot-Savart law diverges for self-induced velocity
- → use local induction approximation (matched asymptotics)
- → Self-induced velocity: $c \approx -\frac{m}{2R}\left(\ln\frac{8R}{r_c} + L_0(m) - 1\right)$,

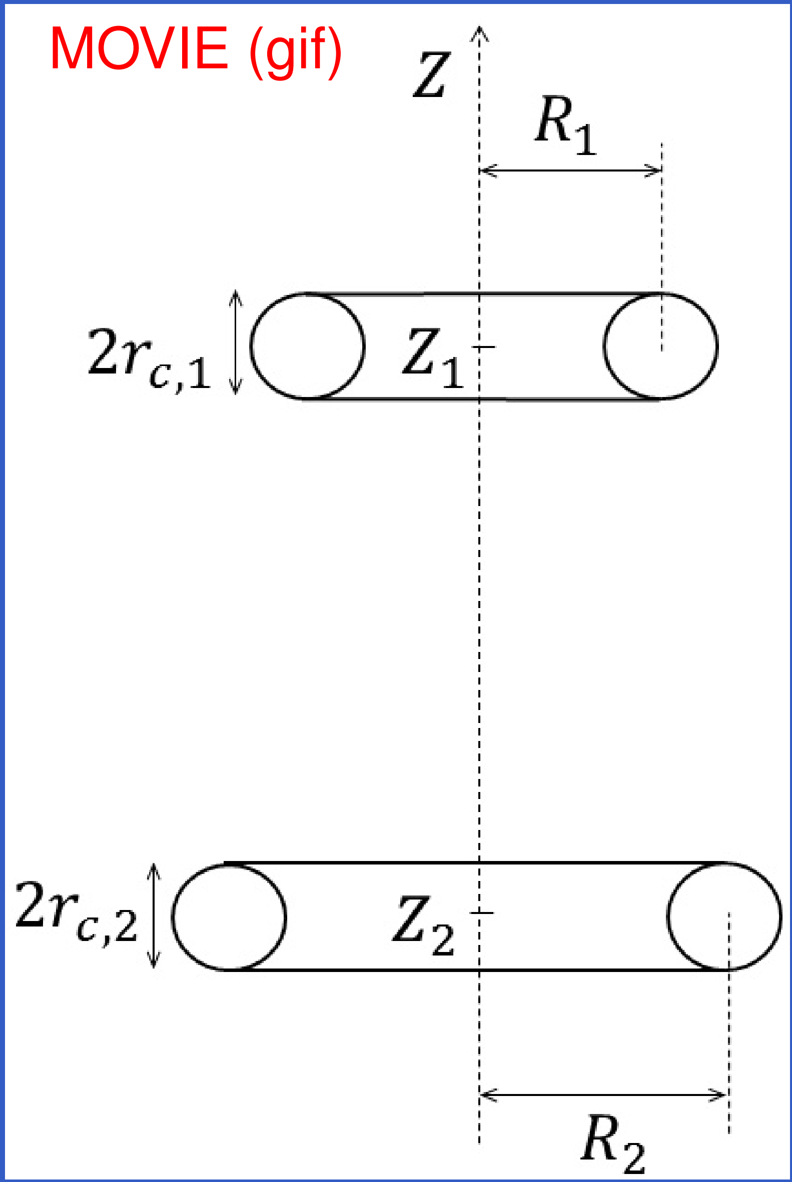
  radius: R, charge: m, core size (healing length): r_c

  $L_0(m) \approx L_0(1)\, m^{-3/2}$: vortex core parameter (numerically)





Particle picture: two VRs ($V_{\text{trap}} = 0$) $\rightarrow$ leapfrogging



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Biot-Savart law & local induction approx:

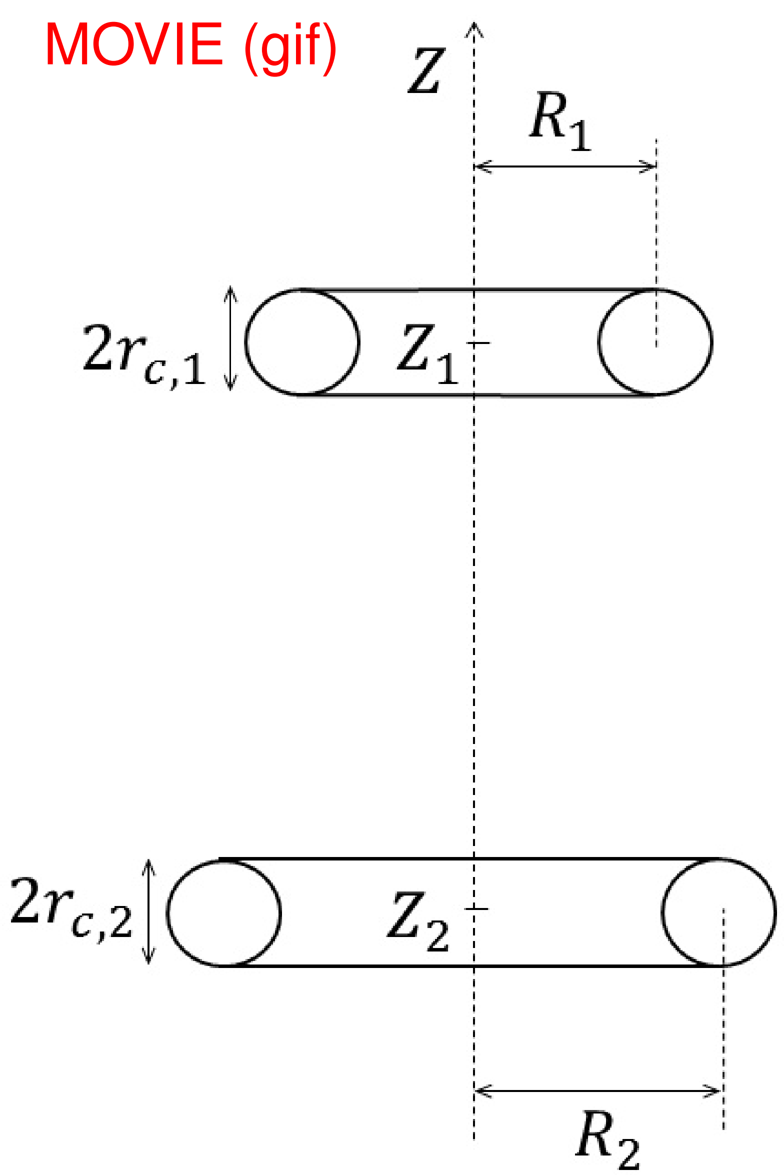
$$\dot{Z}_i = \frac{\kappa_i}{4\pi R_i}\left(\ln\frac{8R_i}{r_{c,i}} - C\right) + \frac{1}{\kappa_i R_i}\frac{\partial U}{\partial R_i},$$
$$\dot{R}_i = -\frac{1}{\kappa_i R_i}\frac{\partial U}{\partial Z_i},$$

where

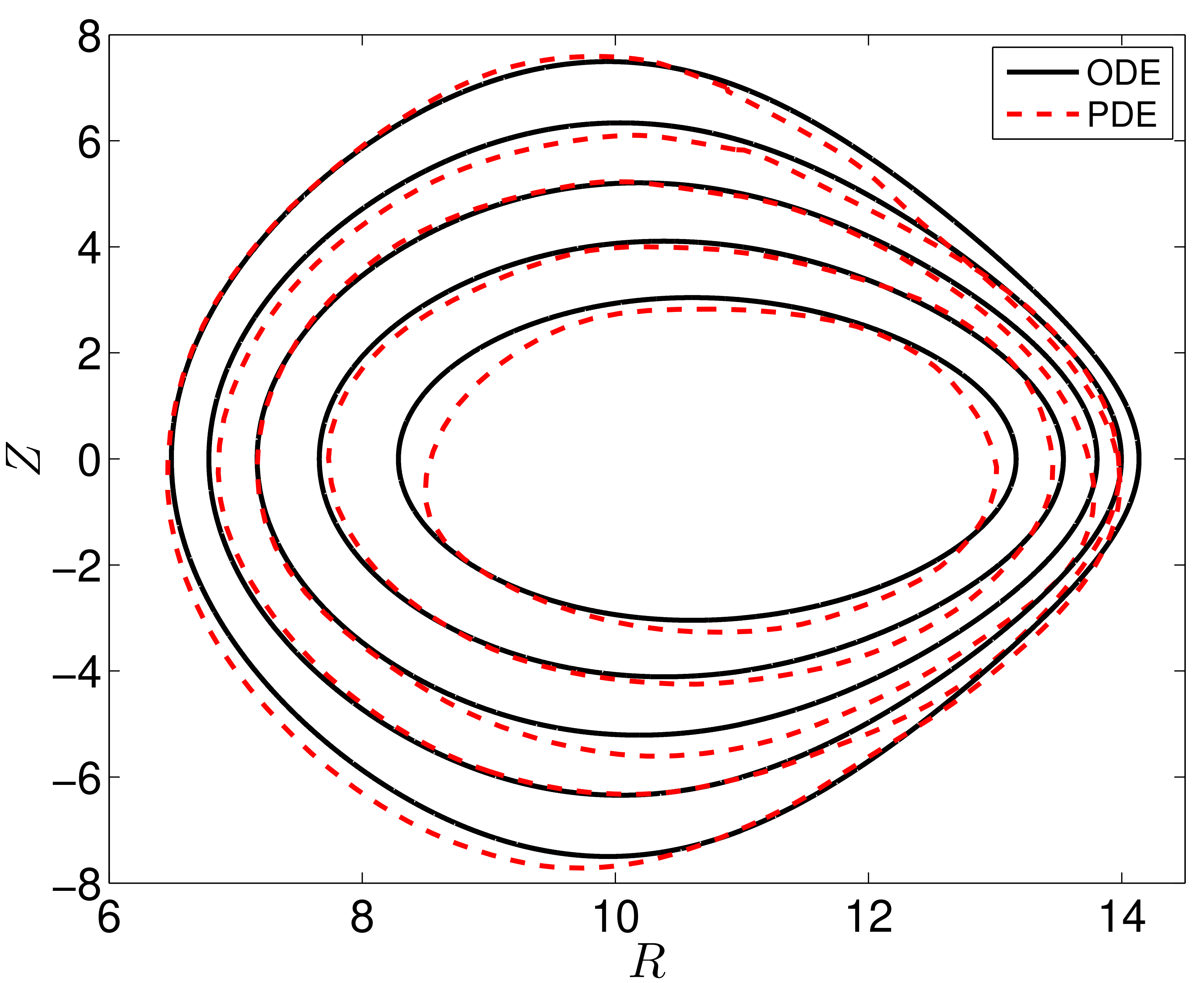
$$U = \frac{1}{2\pi}\sum_{i=1}^{N}\sum_{j>i}^{N}\kappa_i\kappa_j I_{ij},$$

and

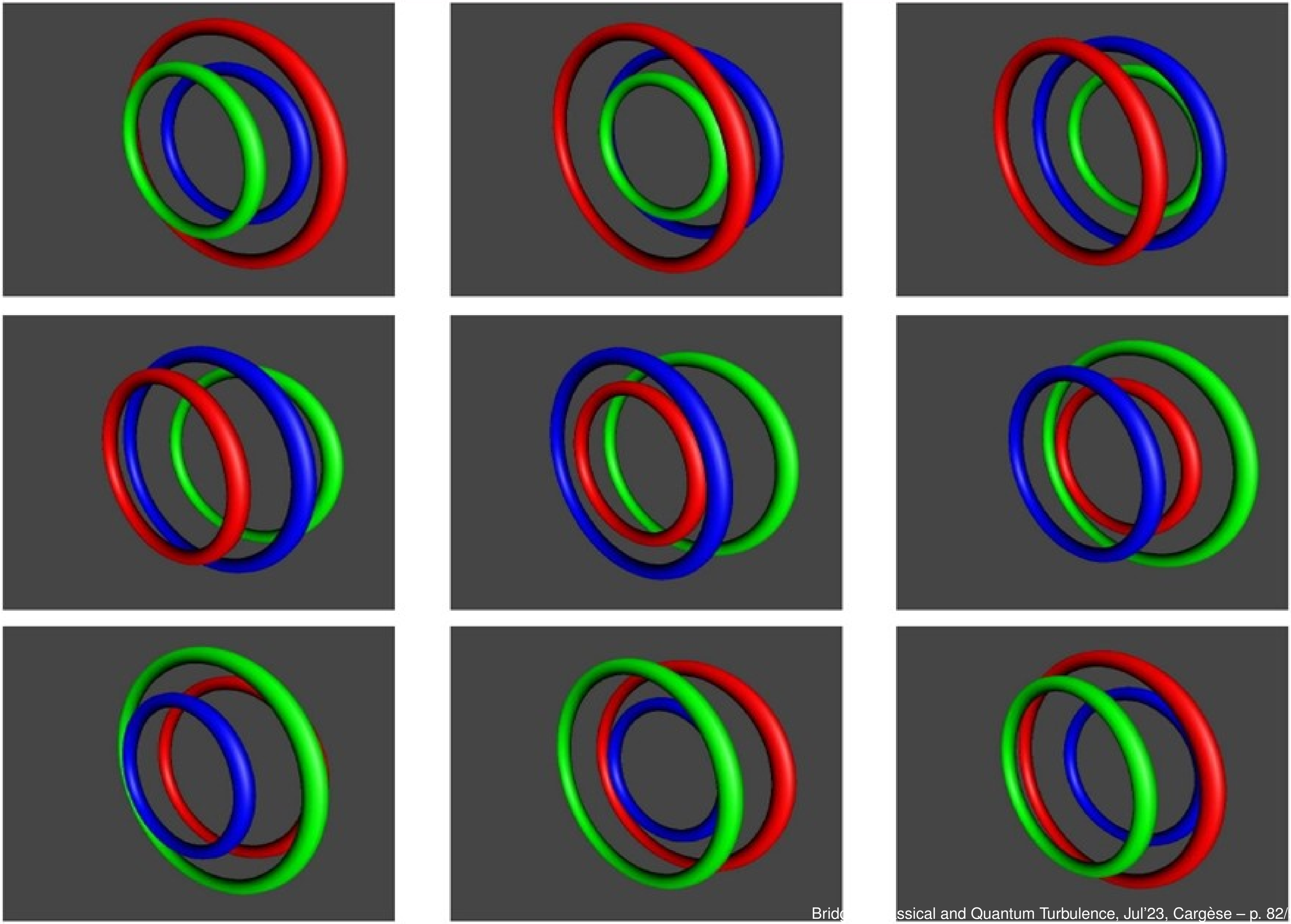
$$I_{ij} = \int_0^\pi \frac{R_i R_j \cos\theta\, d\theta}{\sqrt{(Z_i - Z_j)^2 + R_i^2 + R_j^2 - 2R_iR_j\cos\theta}}$$



Leapfrogging of VRs: PDE vs ODE reduction



Leapfrogging of VR bundles [Barenghi et al., 2014]



Adding $V_{\rm trap}$= harmonic → Particle picture

- naïve approach;
  → just add VR-VR and VR-$V_{\rm trap}$ interactions
  → it does NOT work :(
  - VR-VR interaction (see previous) → H canonical variables (r_i^2, z_i).
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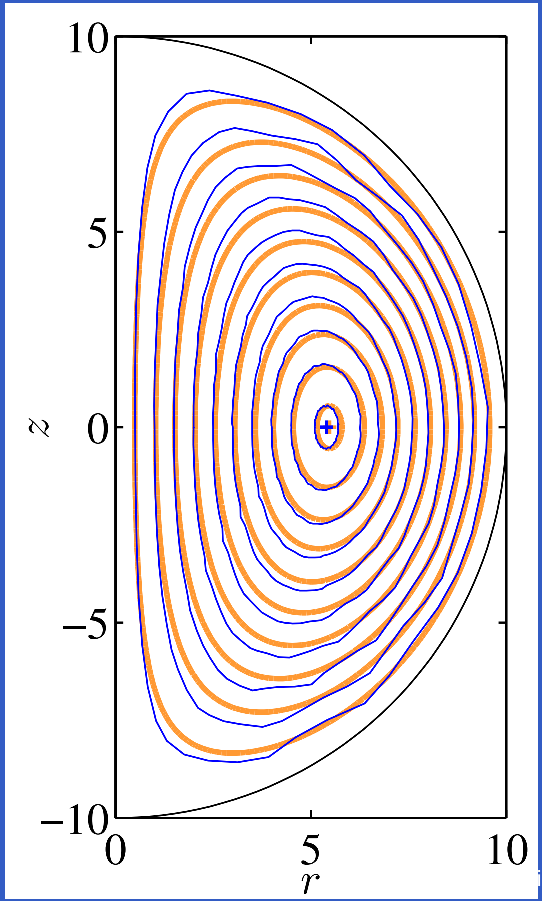

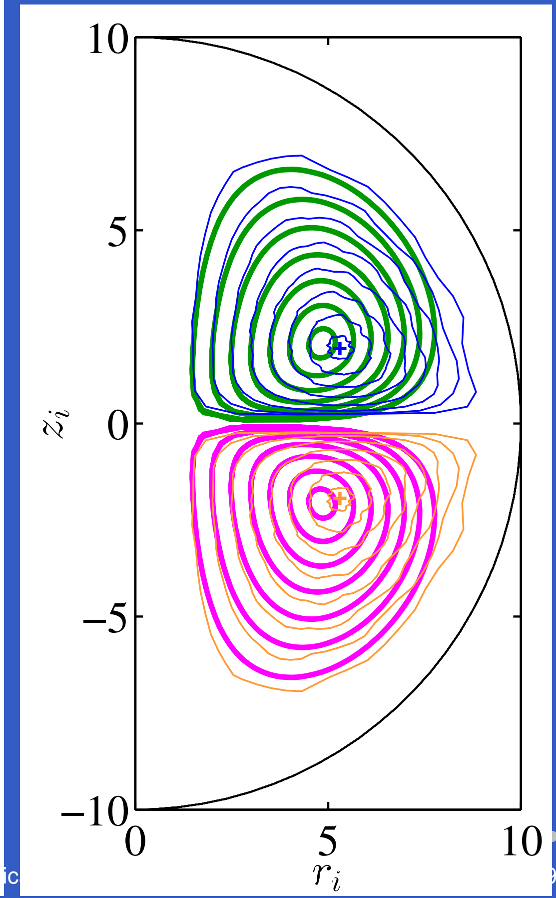

VR scattering scenarios: [Head-on movie] [Scattering movie]

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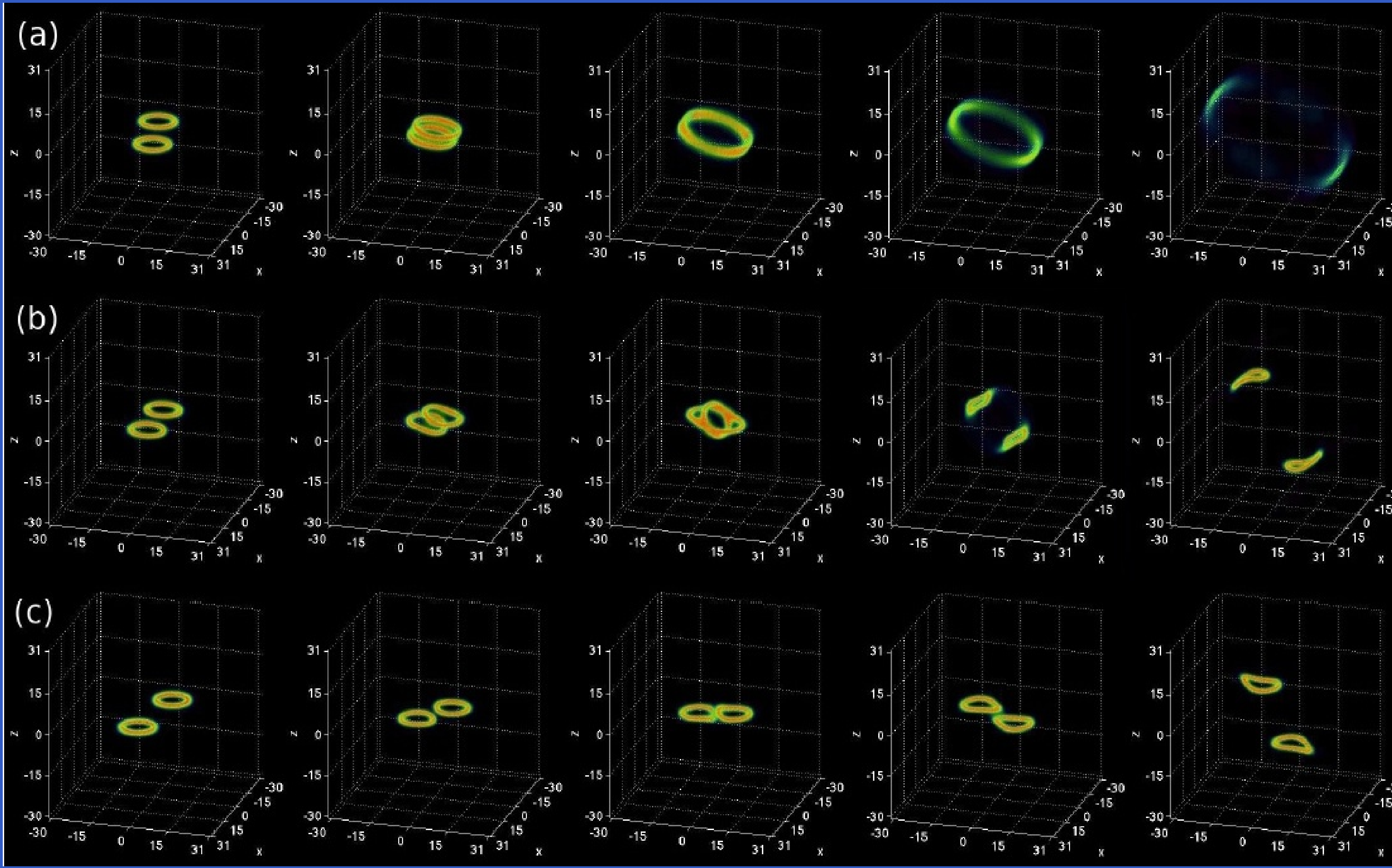
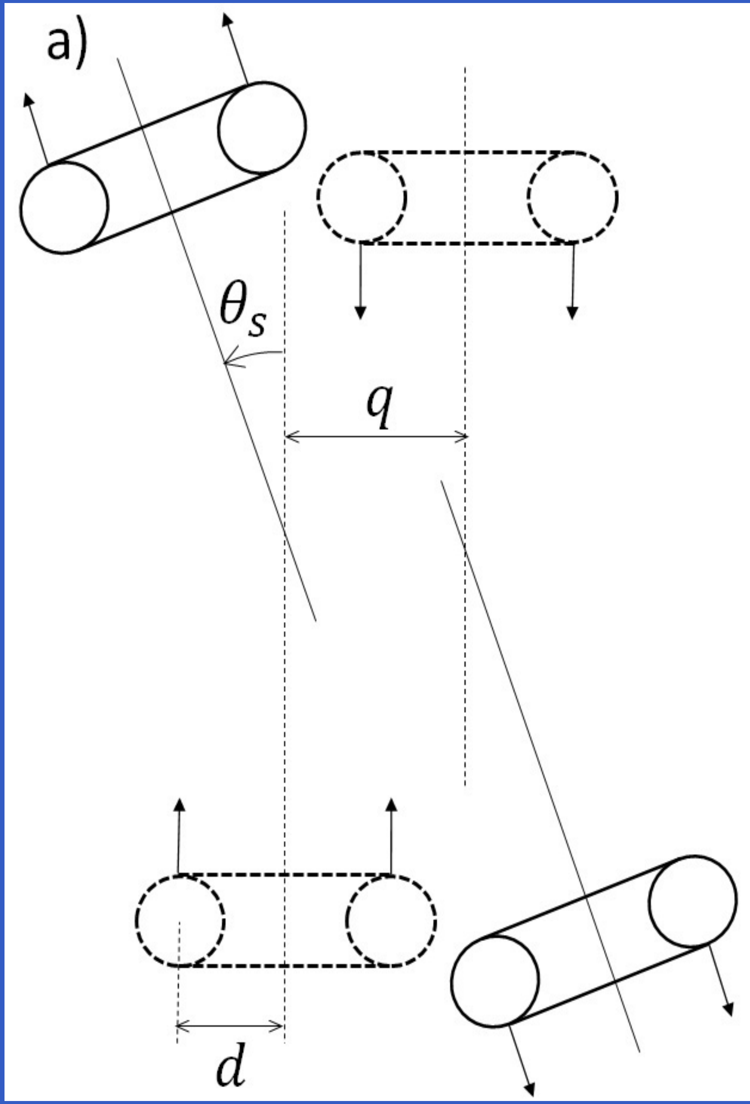
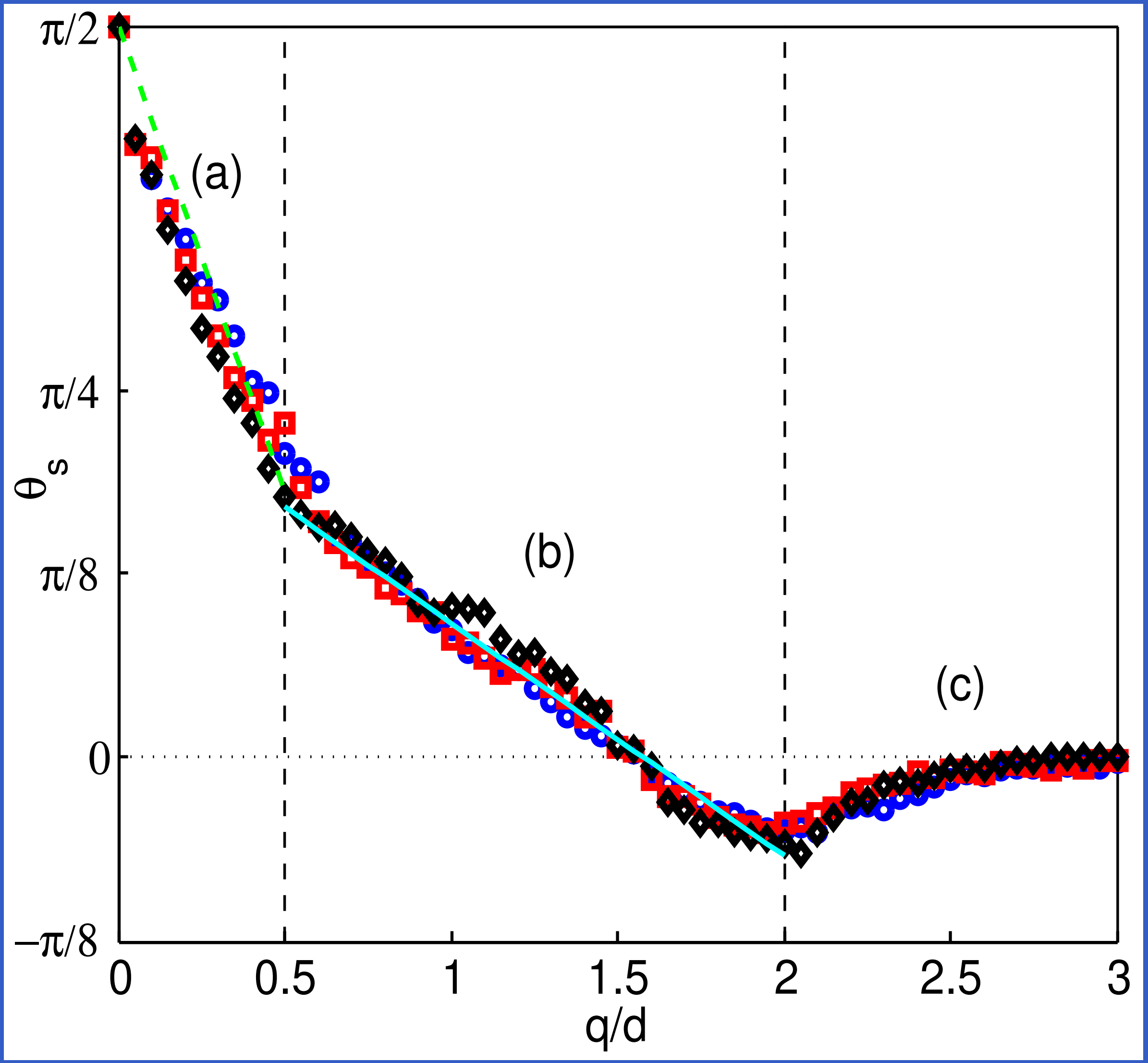


VR scattering angle (numerical results):





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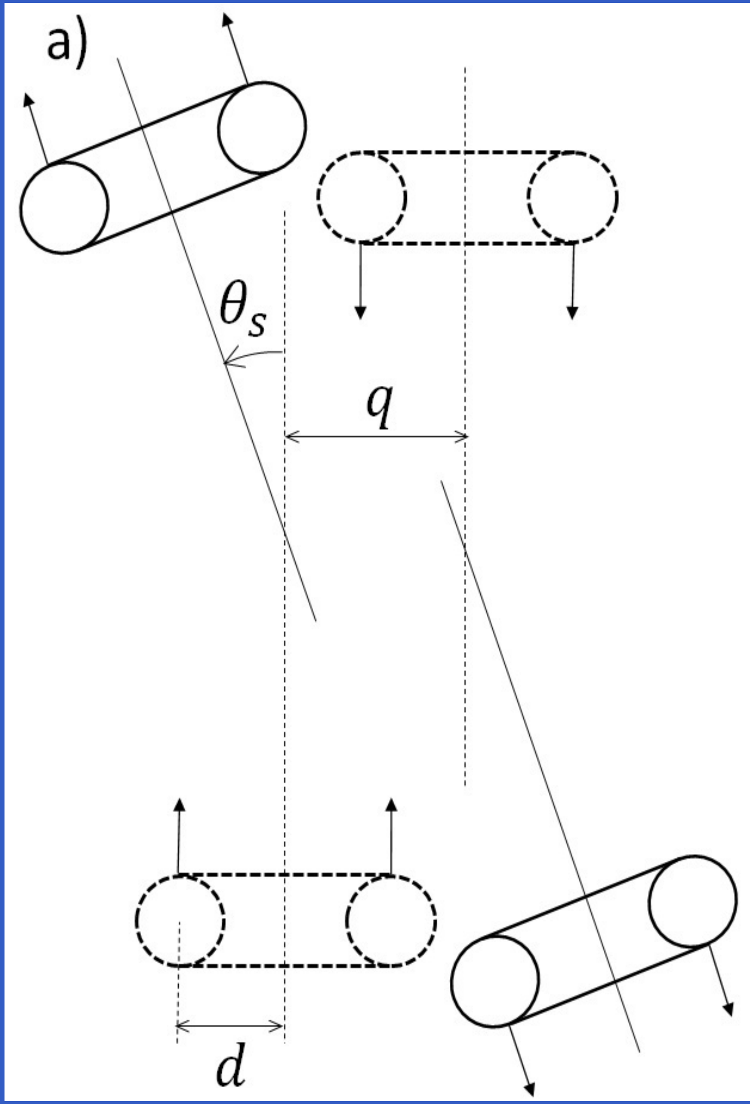

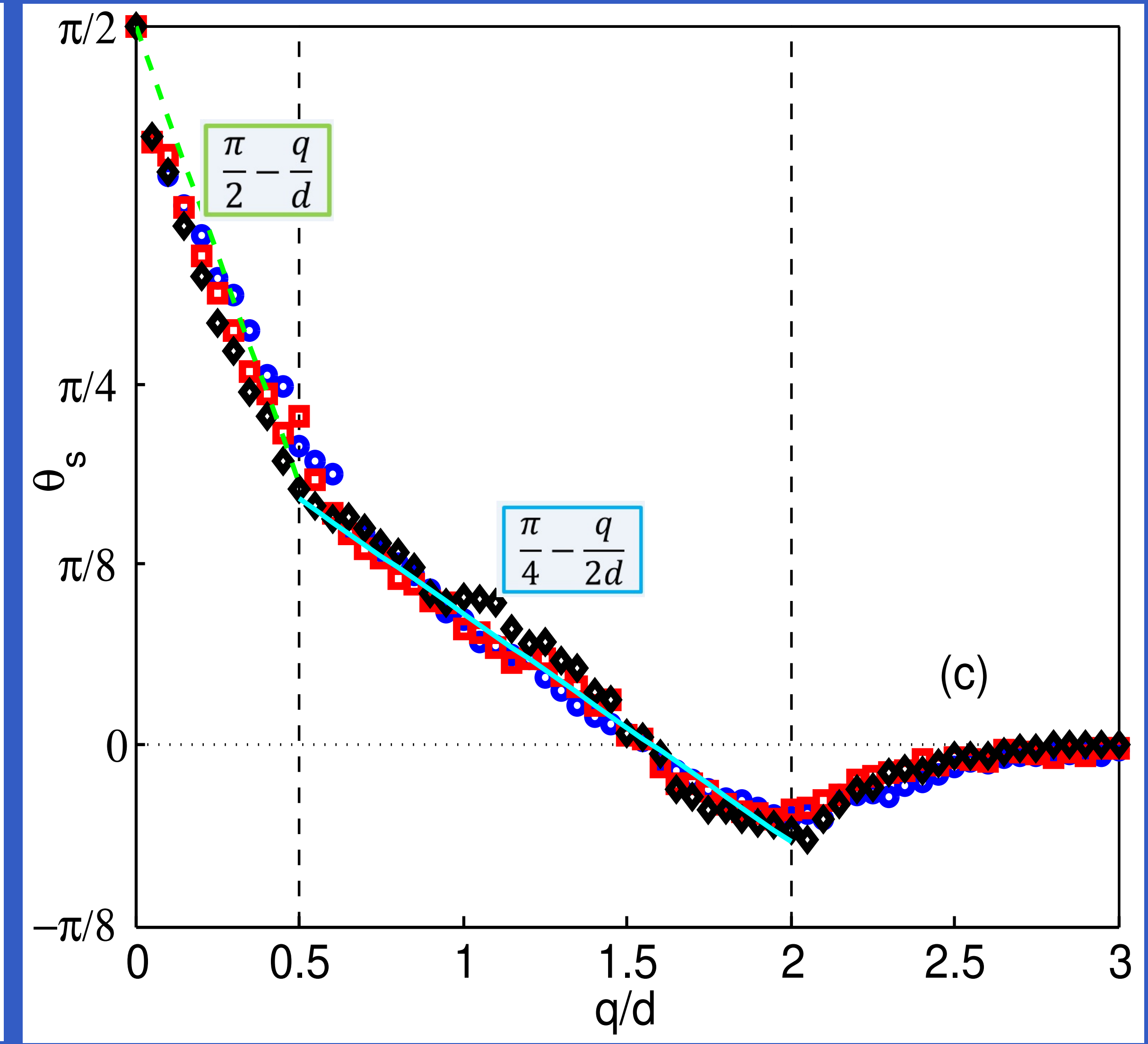

Recap / outlook

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  - Predict types of behavior: bifurcations
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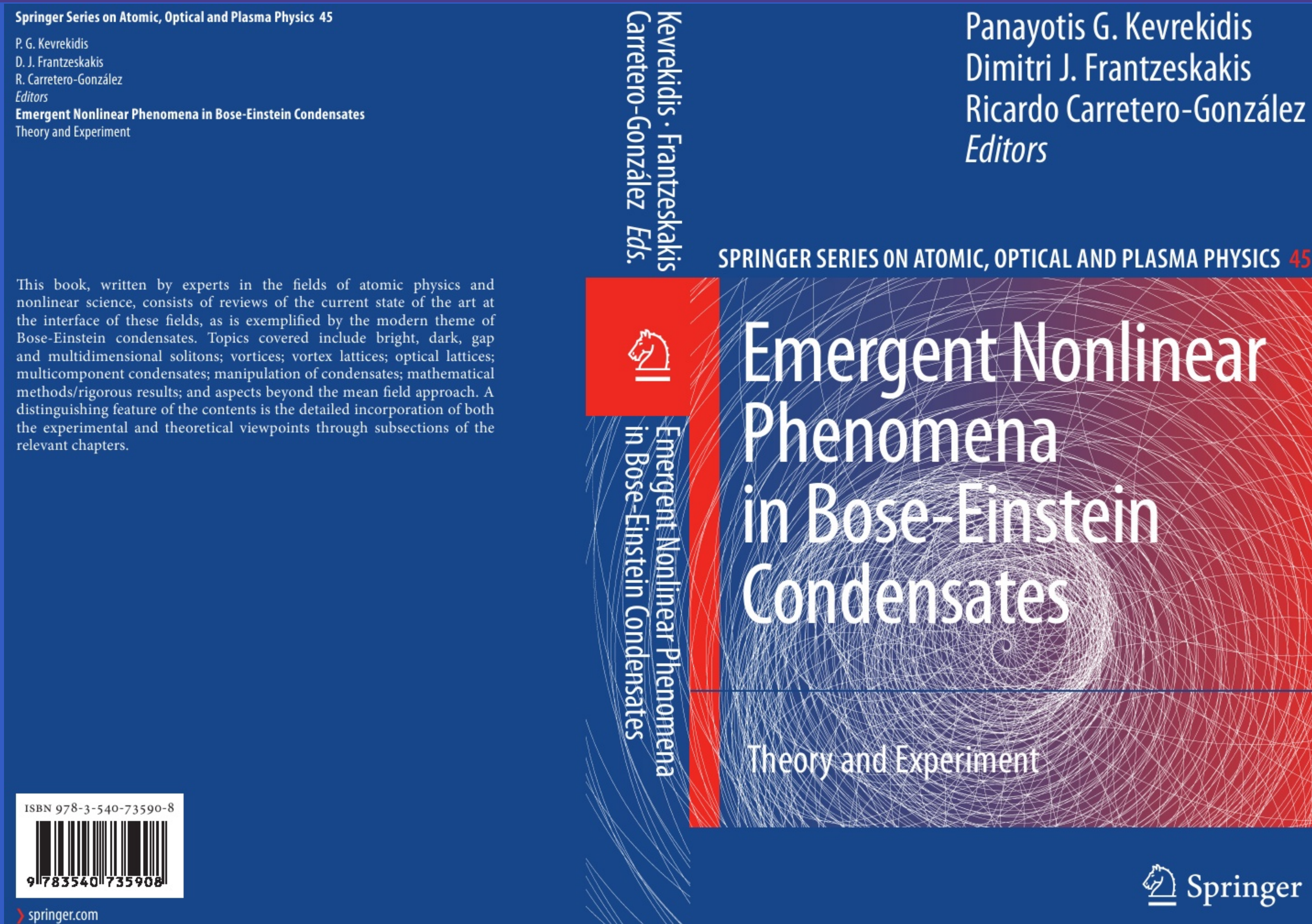
- Outlook and work in progress:
  - Vortices in non-Euclidean geometries (elliptical torus $\to$ chaos!)
  - VR interactions for general angles
  - Introduce internal Kelvin modes ($\to$ changes in self-velocity)
  - Generalizations to two-component BECs
  - Connect with experiments

END ...

Merci !

BECs: Theory and Experiment [Springer, 2008]



DARK Book [SIAM, 2015]

Textbook [OUP, 2023]

Nonlinear Waves & Hamiltonian Systems: From One To Many Degrees of Freedom, From Discrete To Continuum

- Tres amigos:
  - Ricardo Carretero-González
  - Dimitri Frantzeskakis
  - Panos Kevrekidis
- Topics:
  - KdV (inc. KP)
  - Klein-Gordon, Sine-Gordon, ϕ^4 Models
  - NLS (inc. GPE)
  - Discrete Models (KG, DNLS, FPU)
- Target audience: advanced UGs + Grads

NLDS: Nonlinear Dynamical Systems @ SDSU

http://nlds.sdsu.edu/ [Graduate Programs]

MS in Appl. Mathematics with concentration in Dynamical Systems.

- Fall Year 1:
  - MATH-538 : Dynamical Systems & Chaos I
  - MATH-636 : Mathematical Modeling
  - MATH-693A : Numerical Optimization
- Spring Year 1:
  - MATH-531 : Partial Differential Equations
  - MATH-638 : Dynamical Systems & Chaos II
  - MATH-693B : Numerical PDEs
- Fall Year 2:
  - MATH-635 : Pattern Formation
  - MATH-639 : Nonlinear Waves
  - MATH-797 : Research
- Spring Year 2:
  - MATH-799A : Thesis – Project