

# Vortices and vortex rings in quantum superfluids: a quasi-particle approach

Ricardo Carretero

Nonlinear Dynamical Systems Group

<http://nlds.sdsu.edu/>

Computational Science Research Center

Department of Mathematics and Statistics

San Diego State University



# Collaborators/Links (I)



- **Nonlinear Dynamical Systems @ SDSU:** <http://nlds.sdsu.edu/>
  - Peter Blomgren (Numerical PDEs, image processing)
  - Ricardo Carretero (App. math., nonlinear lattices and waves)
  - Chris Curtis (Nonlinear Waves, fluids, optics)
  - Jérôme Gilles (Imaging, wavelets, signal/image processing)
  - Joe Mahaffy (Mathematical biology, delay differential equations)
  - Antonio Palacios (Applied mathematics, bifurcations, symmetries)
  - New hire 2023 (Mathematical Data Science, dynamical systems)



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  - New hire 2023 (Mathematical Data Science, dynamical systems)
- **Research Students involved in nonlinear waves**
  - Miko Stulajter (PhD, Comp. Sci.).
  - Seth Minor, E. Bartley, J. Borjon, (MS, Dyn. Syst.).
  - Recent graduates working on coherent structures: Z. Maches (19), B. Hoang (19), M. Richards (17), J. Rossi (17), J. Talley (17), C. Prieto (16), M. Kandes (16), V. Berardi (15), R. Navarro (13), W. Henry (13), K. Joiner (13), R. Caplan (12), E. Baik (12).

# Collaborators in Nonlinear Waves/Lattices (II)

- David Hall (Amherst Coll.)
- Daniele Sanvitto (Lecce)
- Lorenzo Dominici (Lecce)
- Peter Engels (WSU)
- Brian Anderson (UoA)
- Chiara Daraio (Caltech)
- Lars English (Dickinson)
- Panos Kevrekidis (UMass)
- D. Frantzeskakis (Athens)
- Ionut Danaila (Rouen)
- Stathis Charalampidis (CalPoly)
- Nick Proukakis (Newcastle)
- Tasso Kaper (BU)
- Nate Whitaker (UMass)
- Chris Curtis (SDSU)
- Luis Cisneros-Ake (IPN)
- Wenlong Wang (Chengdu)
- Theo Kolokolnikov (Dalhousie)
- Boris Malomed (Tel Aviv)
- Roy Goodman (NJIT)
- Nathan Kutz (UoW)
- Bernard Deconinck (UoW)
- Dan Spirn (Minnesota)
- Mason Porter (UCLA)
- Yuri Kivshar (Cambera)
- W. Królikowski (TAMU-Qatar)
- Jesús Cuevas (Sevilla)
- Peter Schmelcher (Heidelberg)
- Klaus Sengstock (Hamburg)
- Vladimir Konotop (Lisboa)
- Pedro Torres (Granada)
- Todd Kapitula (Calvin Col.)
- Keith Promislow (MSU)
- Bernard Deconinck (UoW)
- Lincoln Carr (Col. Sch. Mines)
- Ashton Bradley (Otago)
- Jared Bronski (UI-UC)
- etc, ... (>150)

# Road map

- Introduction
  - BECs, External trapping → controlling dimensionality
  - Motivation: physical experiments

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  - Vortex dipoles (+, -) → the Spirograph
  - Vortex pairs (+, +) → Bifurcation analysis
  - Bifurcation in the actual experiment!!!
  - Dissipation → polaritonic BECs
  - Anisotropy → complex periodic orbits
  - Non-Eucliden geometries

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- Going 3D → Vortex rings (VRs)
  - VR → self-induced velocity
  - VR-VR interactions → Biot-Savart law
  - Leapfrogging dynamics of VRs
  - VR scattering (co-planar)

# Gross-Pitaevskii Eq.:

Close to  $T = 0$  BEC  $\rightarrow$  Gross-Pitaevskii Eq.:

$$i\hbar \frac{\partial \psi}{\partial t} = \left[ -\frac{\hbar^2}{2m} \nabla^2 + V_{\text{ext}}(\mathbf{r}) + g |\psi|^2 \right] \psi,$$

- $\psi(x, y, z, t)$ : wavefunction,  
 $|\psi|^2$ : density,  
 $N = \iiint |\psi|^2 dV$ : # of atoms,
- nonlinear coeff:  $g = 4\pi\hbar^2 a_s / m$ , where  $a_s$  scattering length:
  - $a_s > 0$  : repulsive : ( $^{23}\text{Na}$ ,  $^{87}\text{Rb}$ , H,  $^4\text{He}$ ,  $^{85}\text{Rb}$ )  $\rightarrow$  [DSs, vortices, VRs, VLs]
  - $a_s < 0$  : attractive : ( $^7\text{Li}$ ,  $^{85}\text{Rb}$ )  $\rightarrow$  [BSs, Bose Nova]



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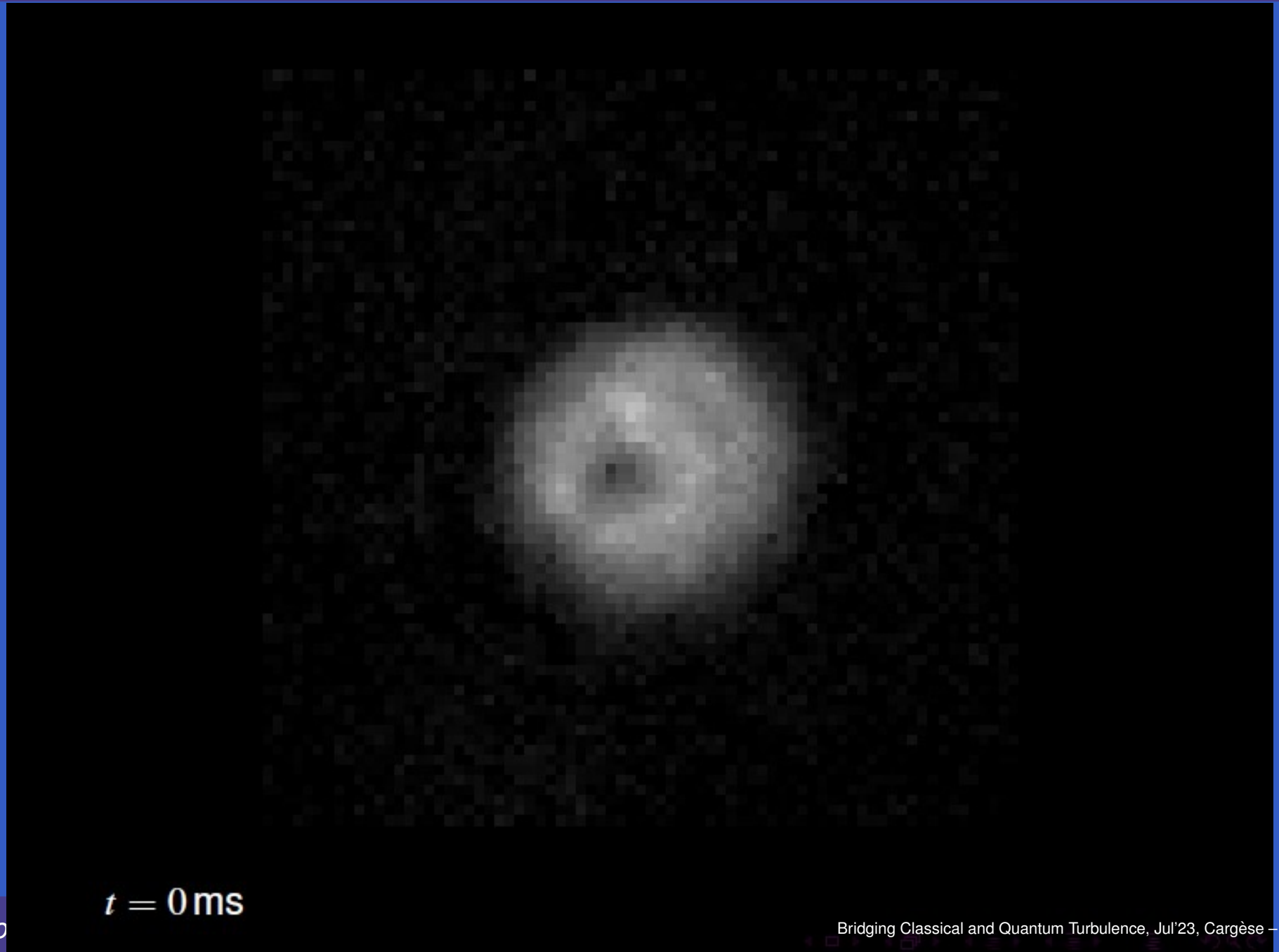
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  - $a_s < 0$  : attractive : ( $^7\text{Li}$ ,  $^{85}\text{Rb}$ )  $\rightarrow$  [BSs, Bose Nova]
- External potential:  $V_{\text{ext}}(\mathbf{r}) = \frac{1}{2}m\omega_r^2 r^2 + \frac{1}{2}m\omega_z^2 z^2$ 
  - $\omega_r \gg \omega_z \rightarrow$  1D BEC  $\rightarrow$  bright/dark solitons
  - $\omega_z \gg \omega_r \rightarrow$  2D BEC  $\rightarrow$  vortices
  - $\omega_r \approx \omega_z \rightarrow$  3D BEC  $\rightarrow$  vortex lines/vortex rings

# Vortex dynamics in EXPERIMENTAL BECs (David Hall's experiment)

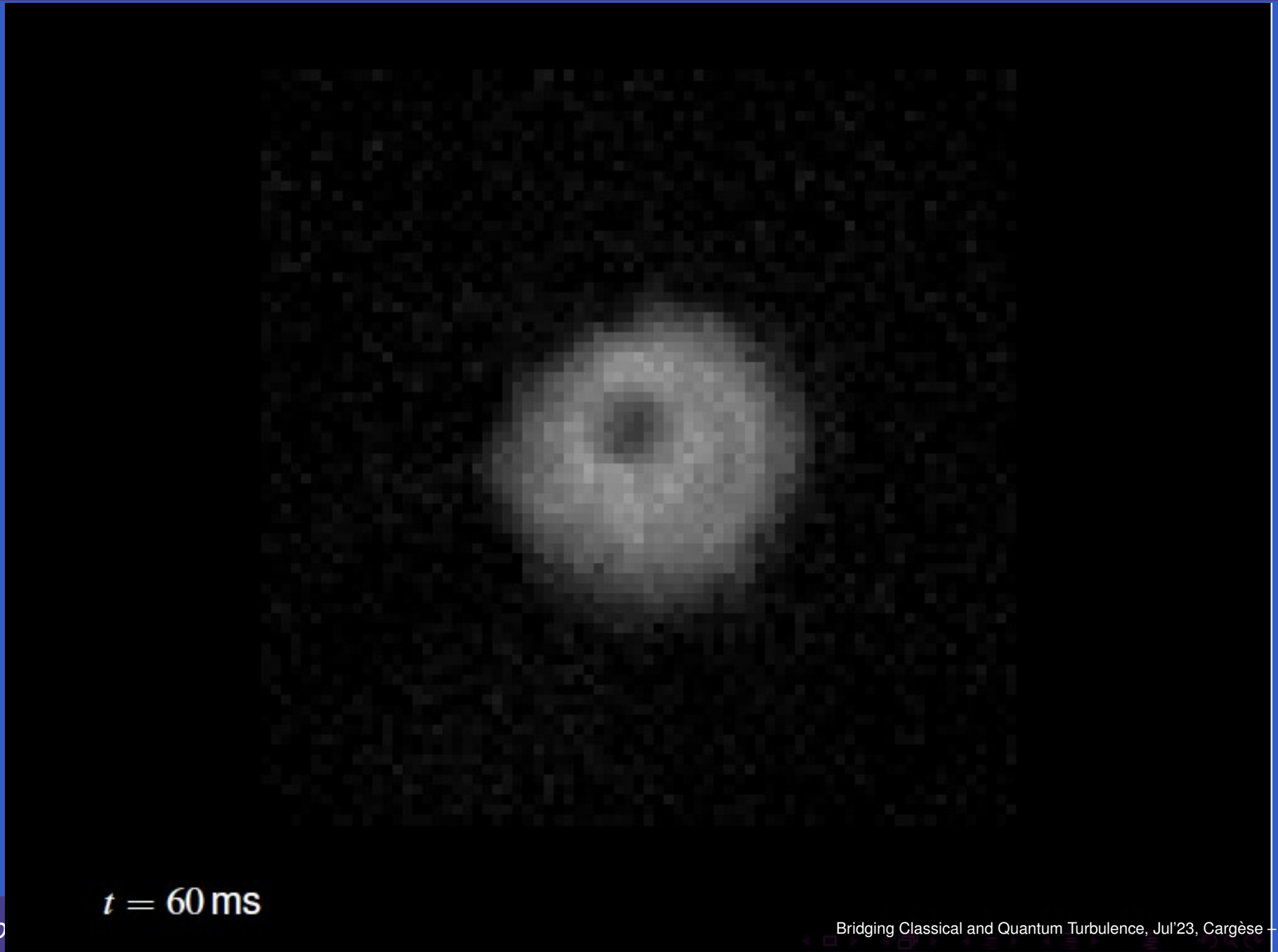
# Vortex precession

# Experimental BEC vortices: precession in MT

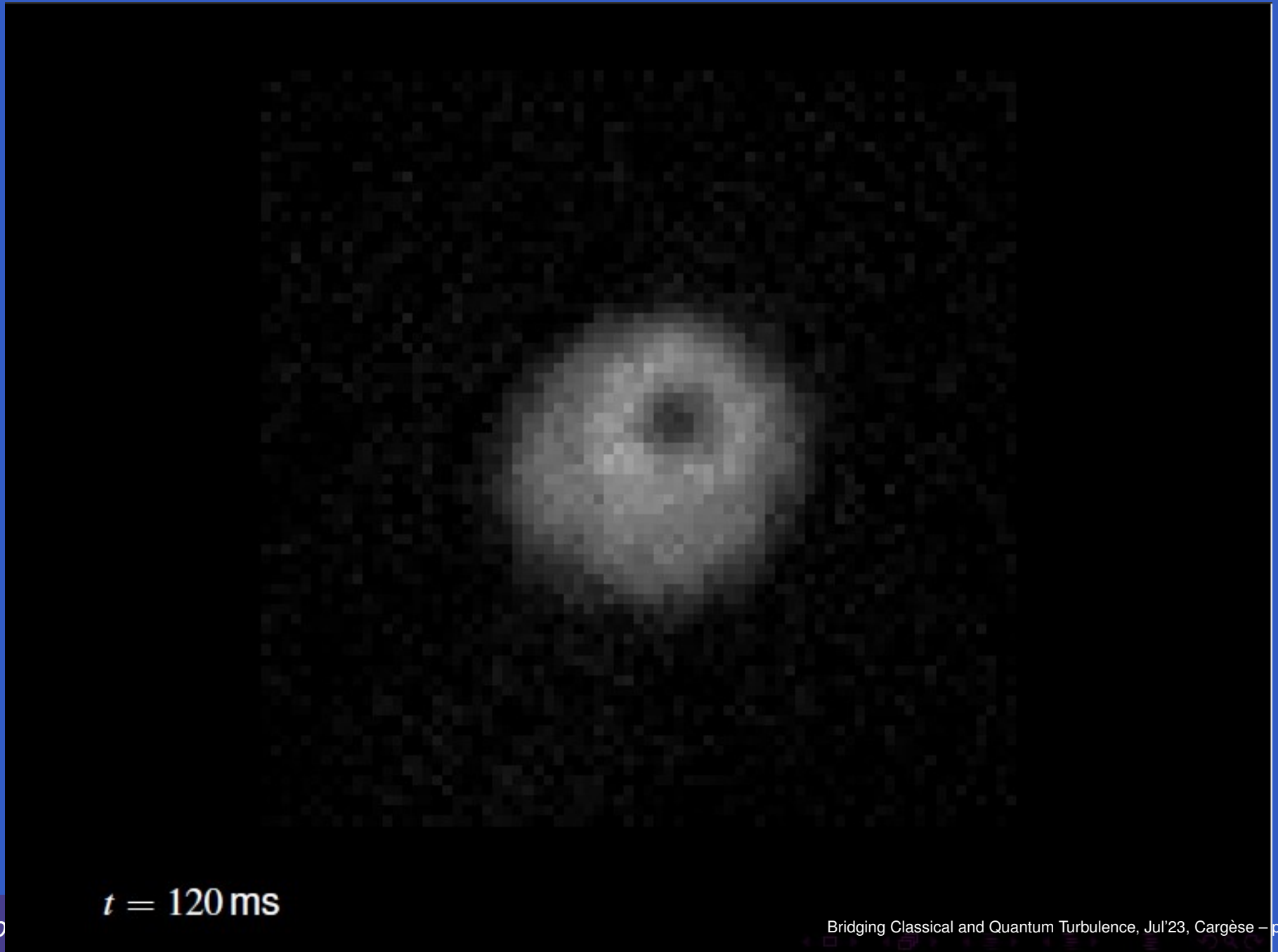


$t = 0 \text{ ms}$

# Experimental BEC vortices: precession in MT

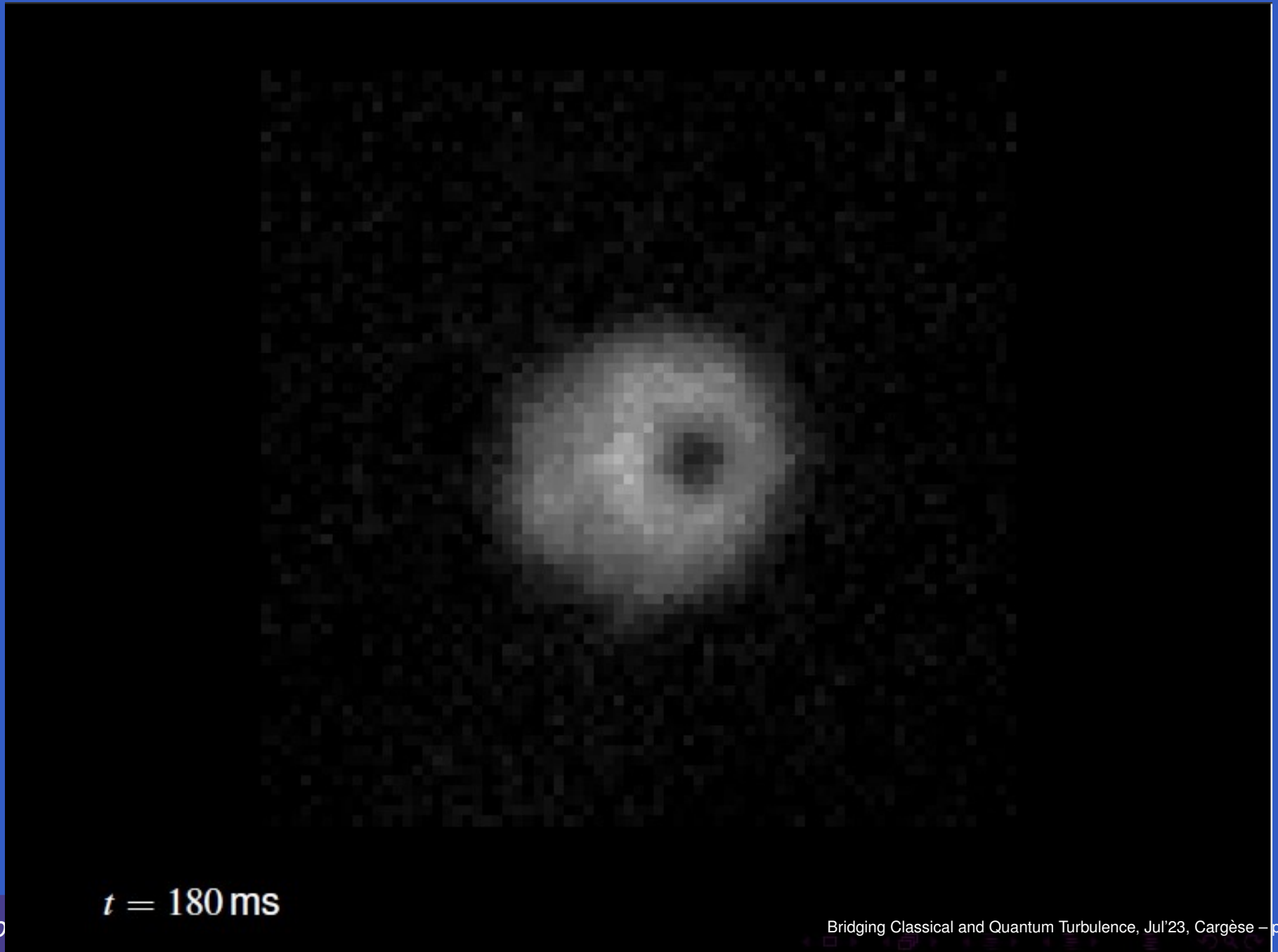


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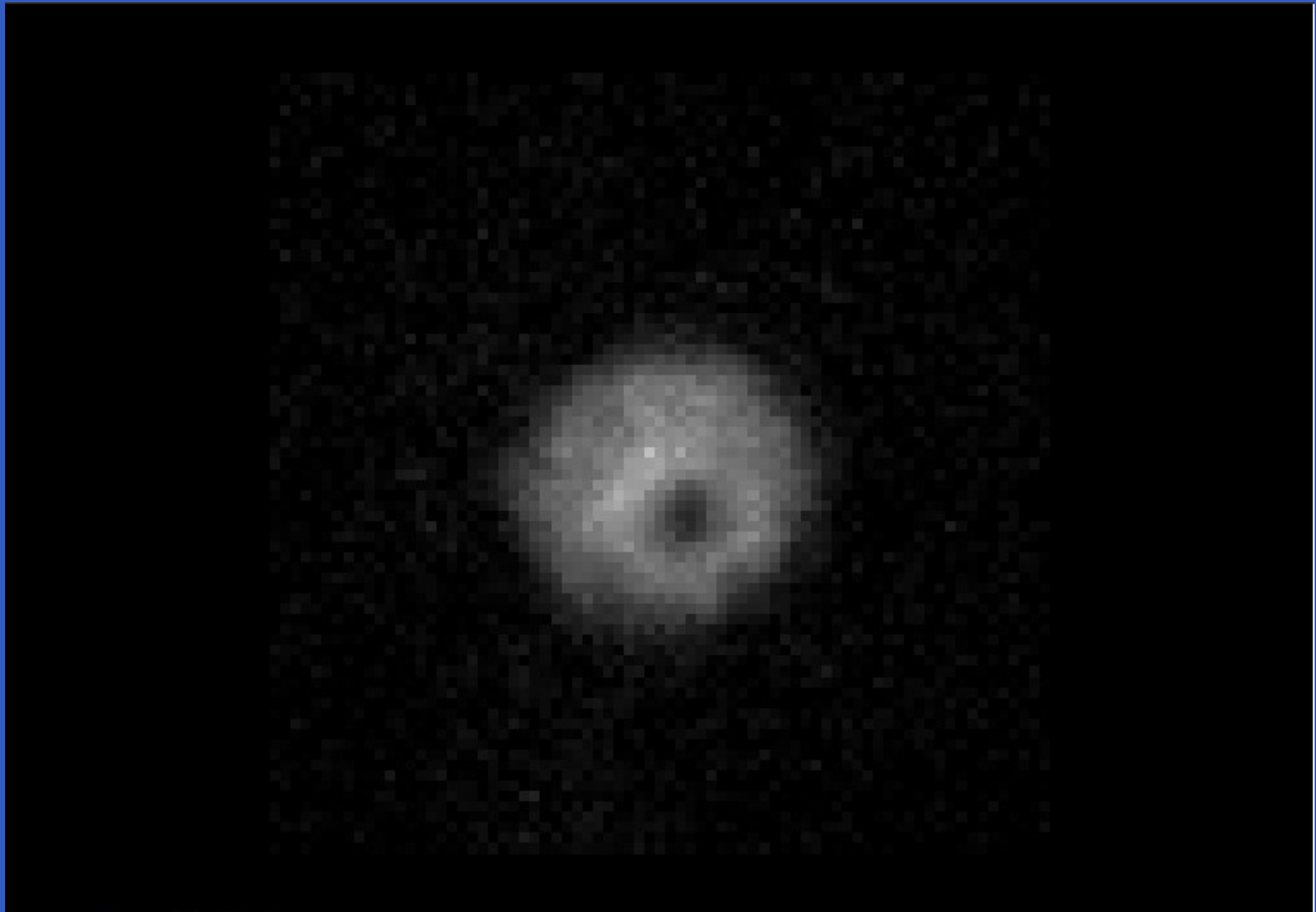
$t = 120$  ms

# Experimental BEC vortices: precession in MT



$t = 180$  ms

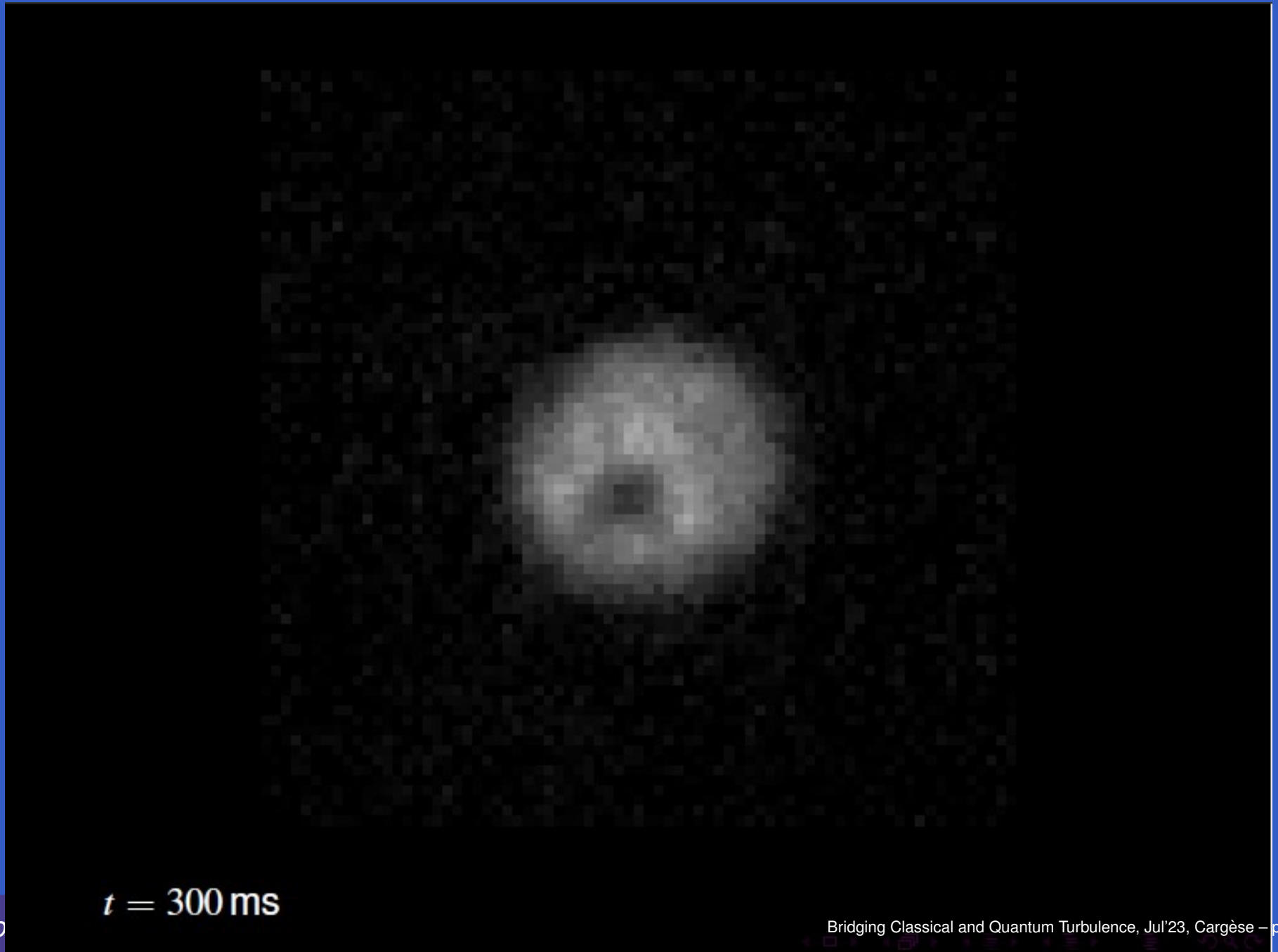
# Experimental BEC vortices: precession in MT



$t = 240 \text{ ms}$

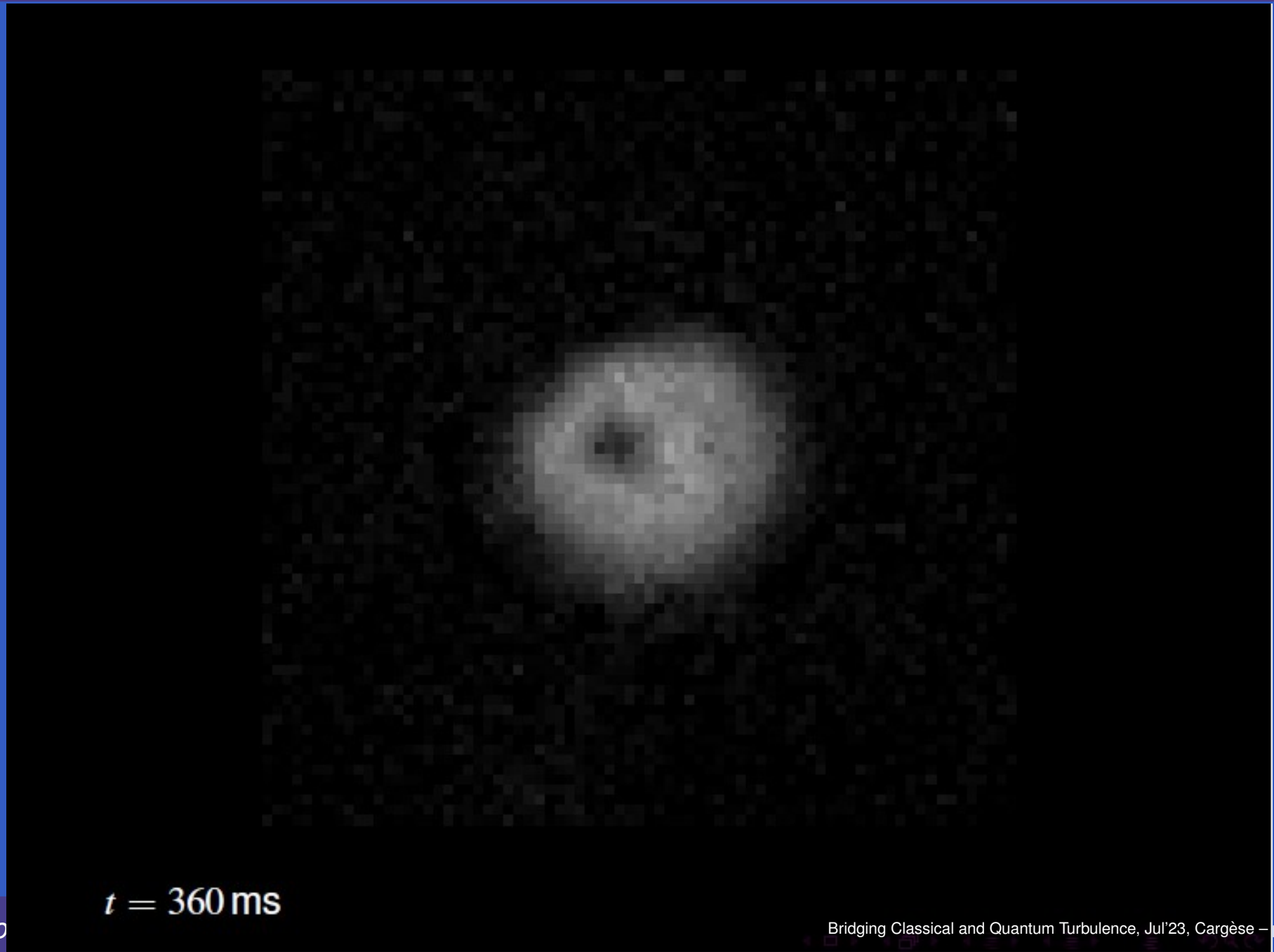


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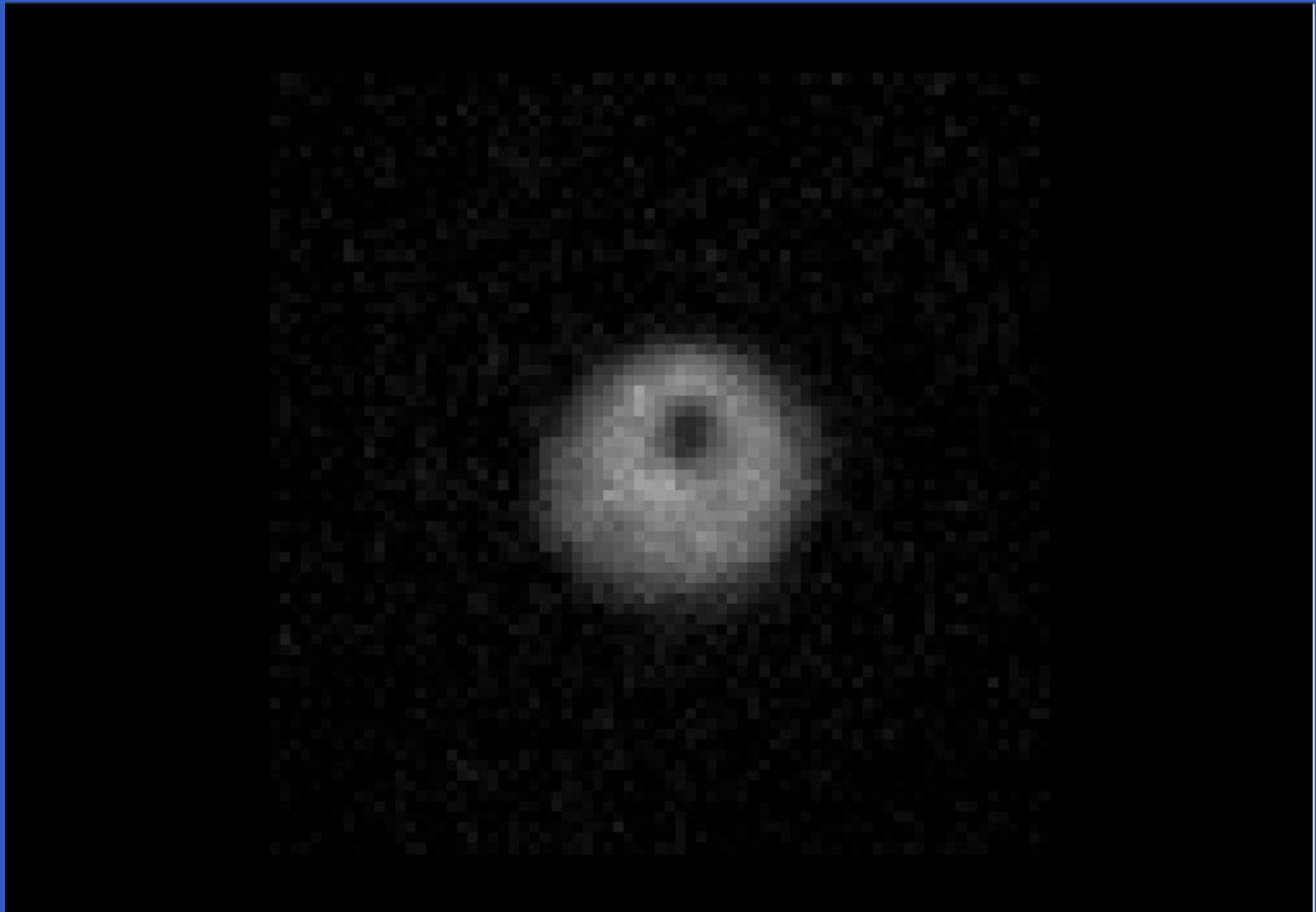
$t = 300 \text{ ms}$

# Experimental BEC vortices: precession in MT



$t = 360 \text{ ms}$

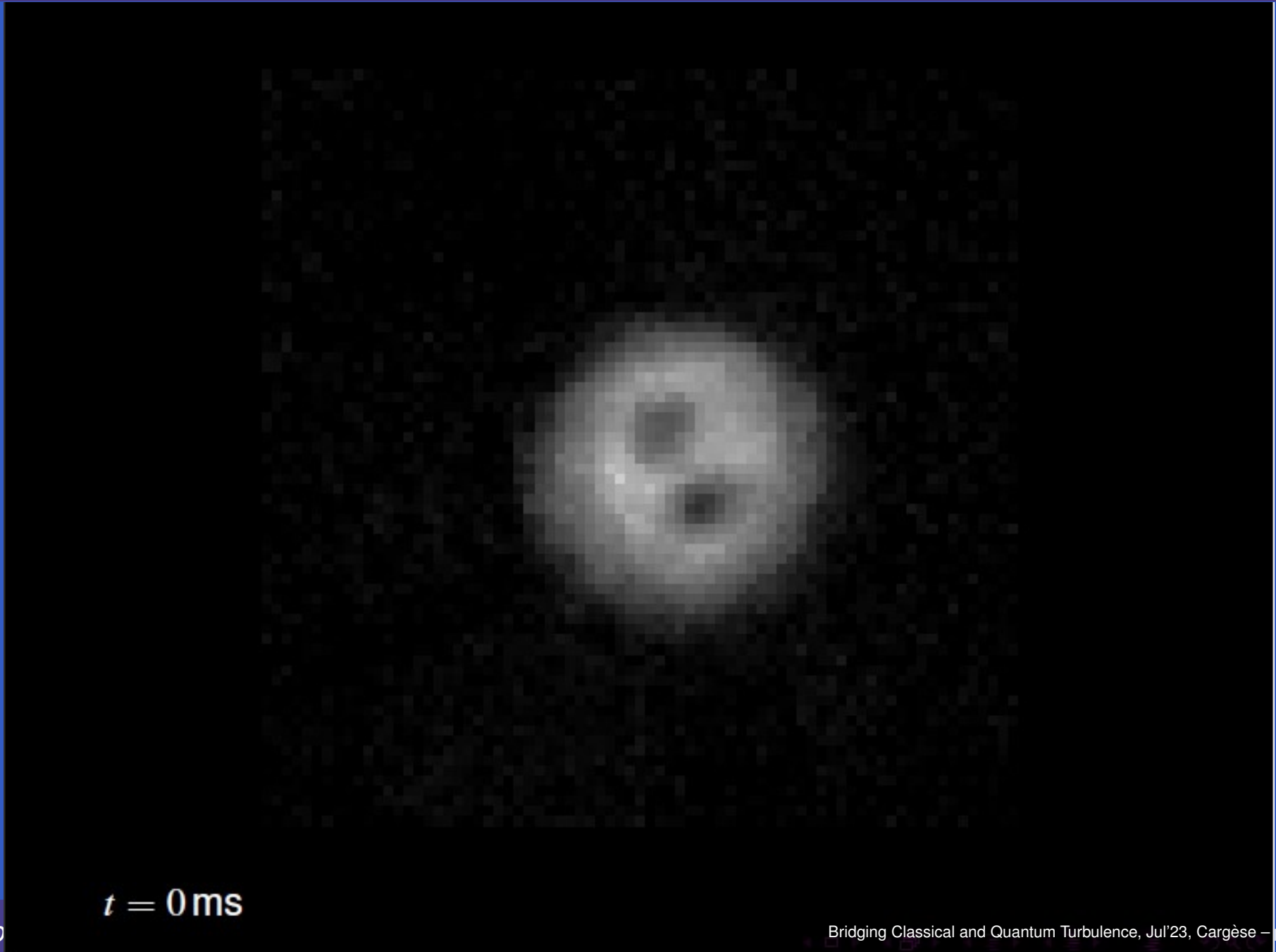
# Experimental BEC vortices: precession in MT



$t = 420$  ms

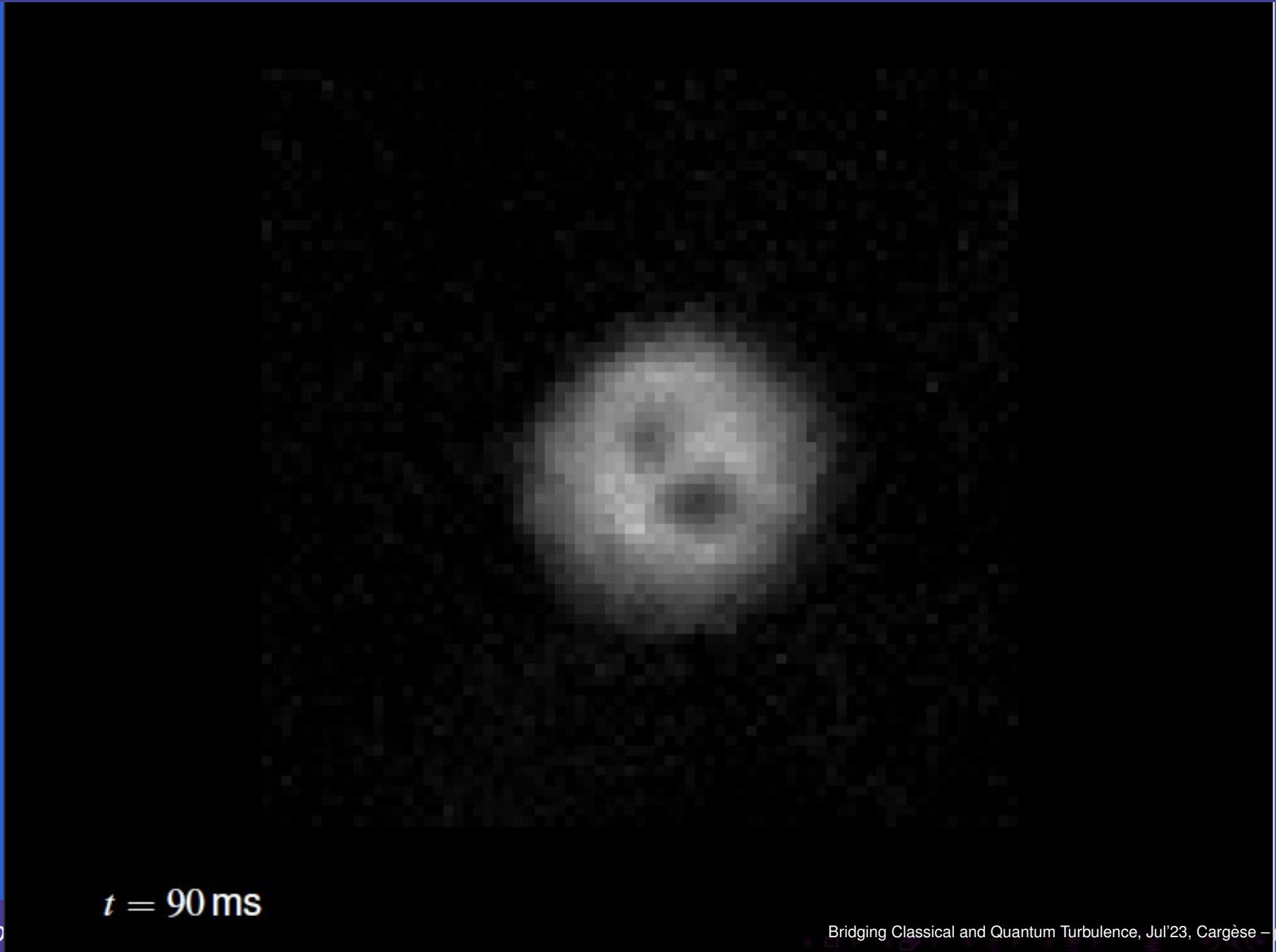
# Static vortex dipole

# Experimental BEC vortices: static vortex dipole



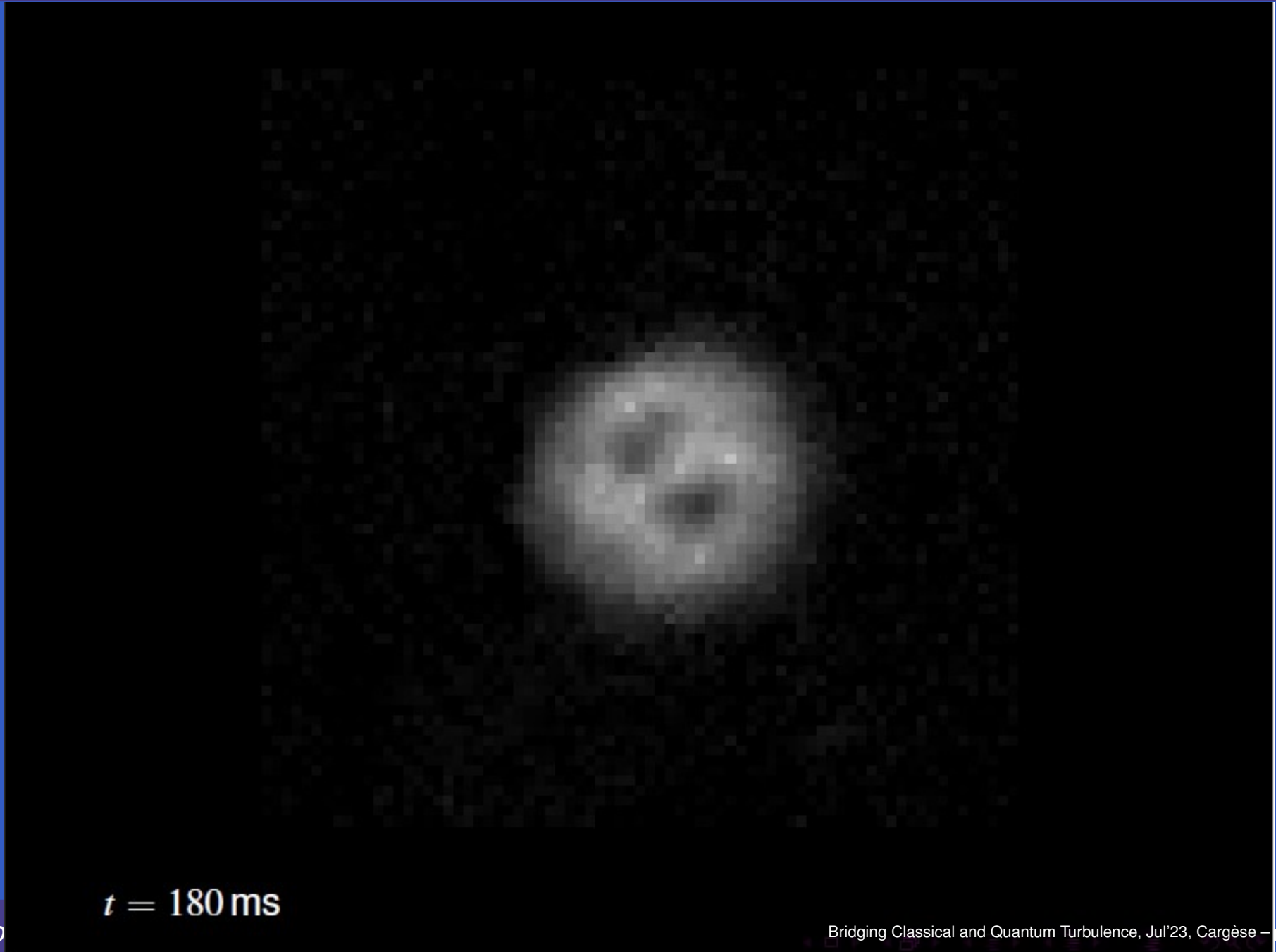
$t = 0 \text{ ms}$

# Experimental BEC vortices: static vortex dipole



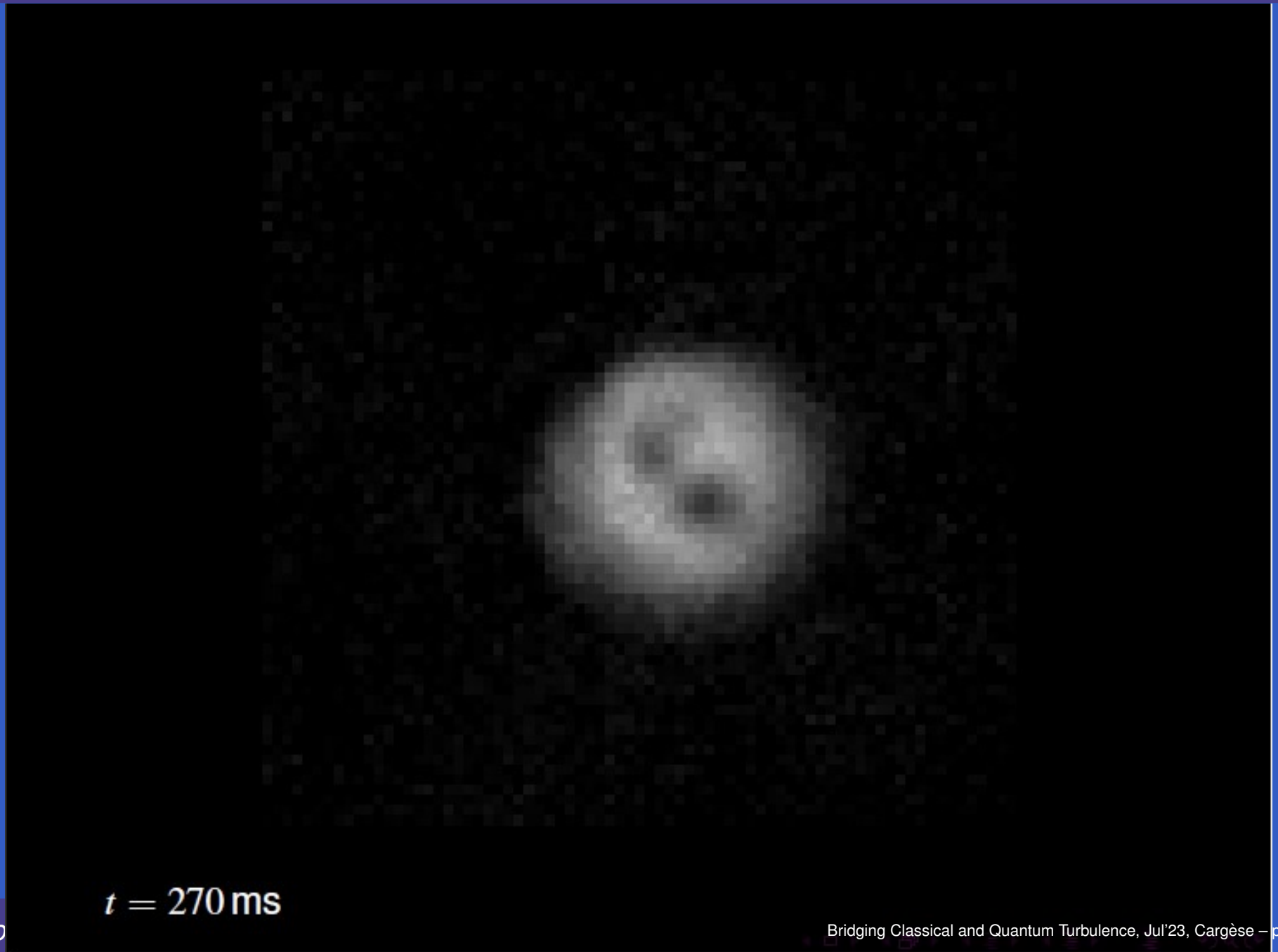
$t = 90 \text{ ms}$

# Experimental BEC vortices: static vortex dipole



$t = 180 \text{ ms}$

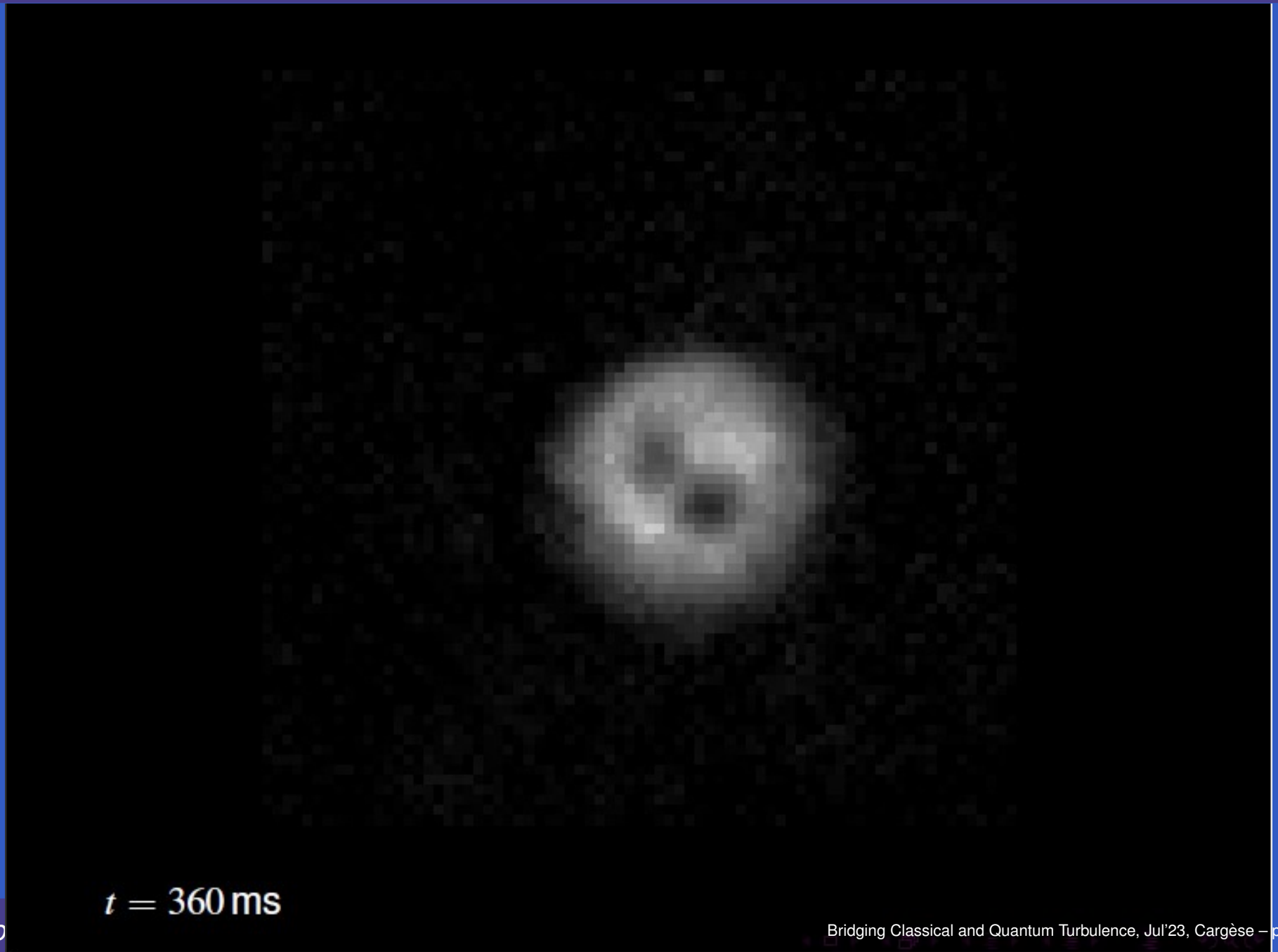
# Experimental BEC vortices: static vortex dipole



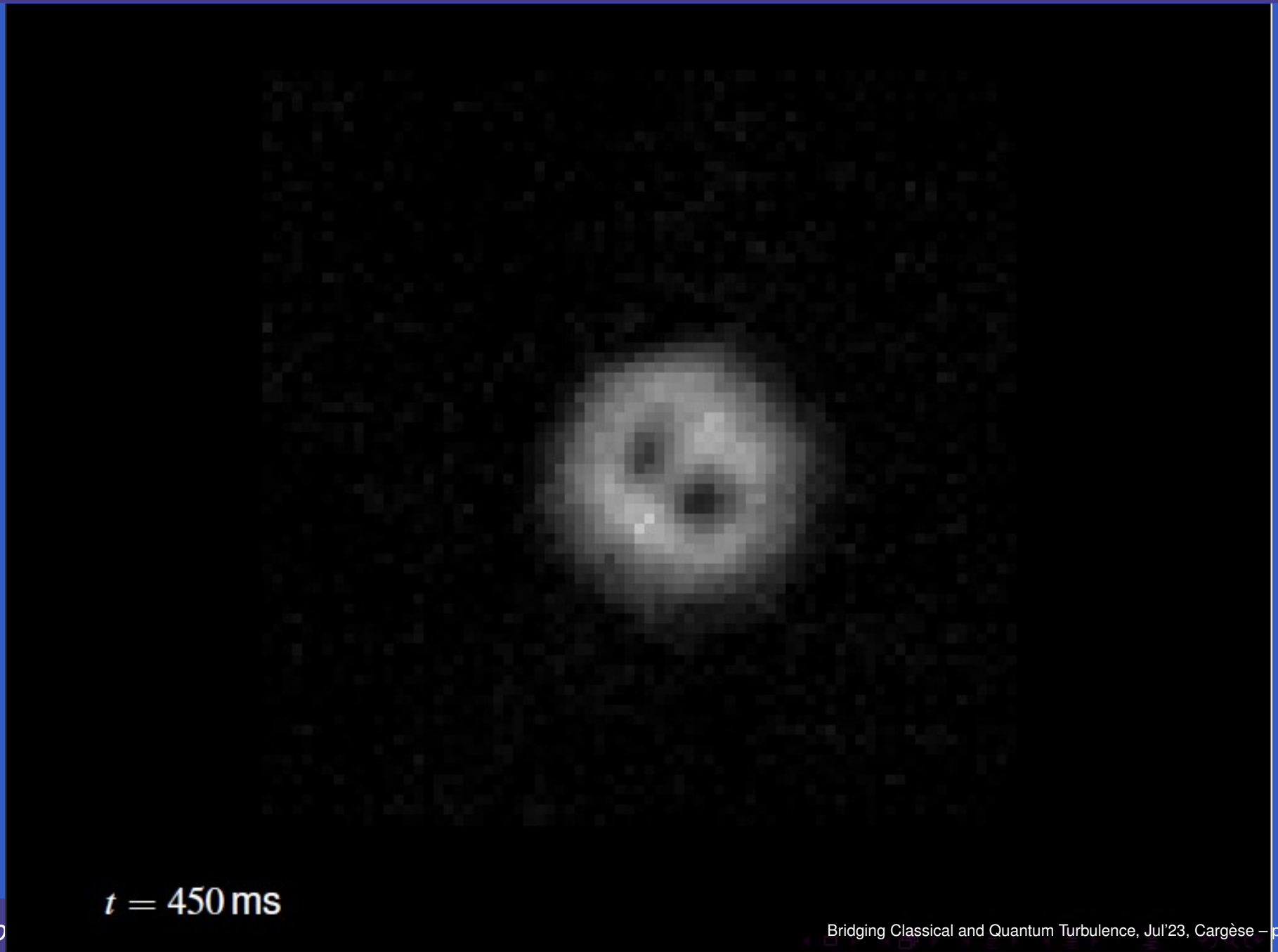
$t = 270 \text{ ms}$



# Experimental BEC vortices: static vortex dipole

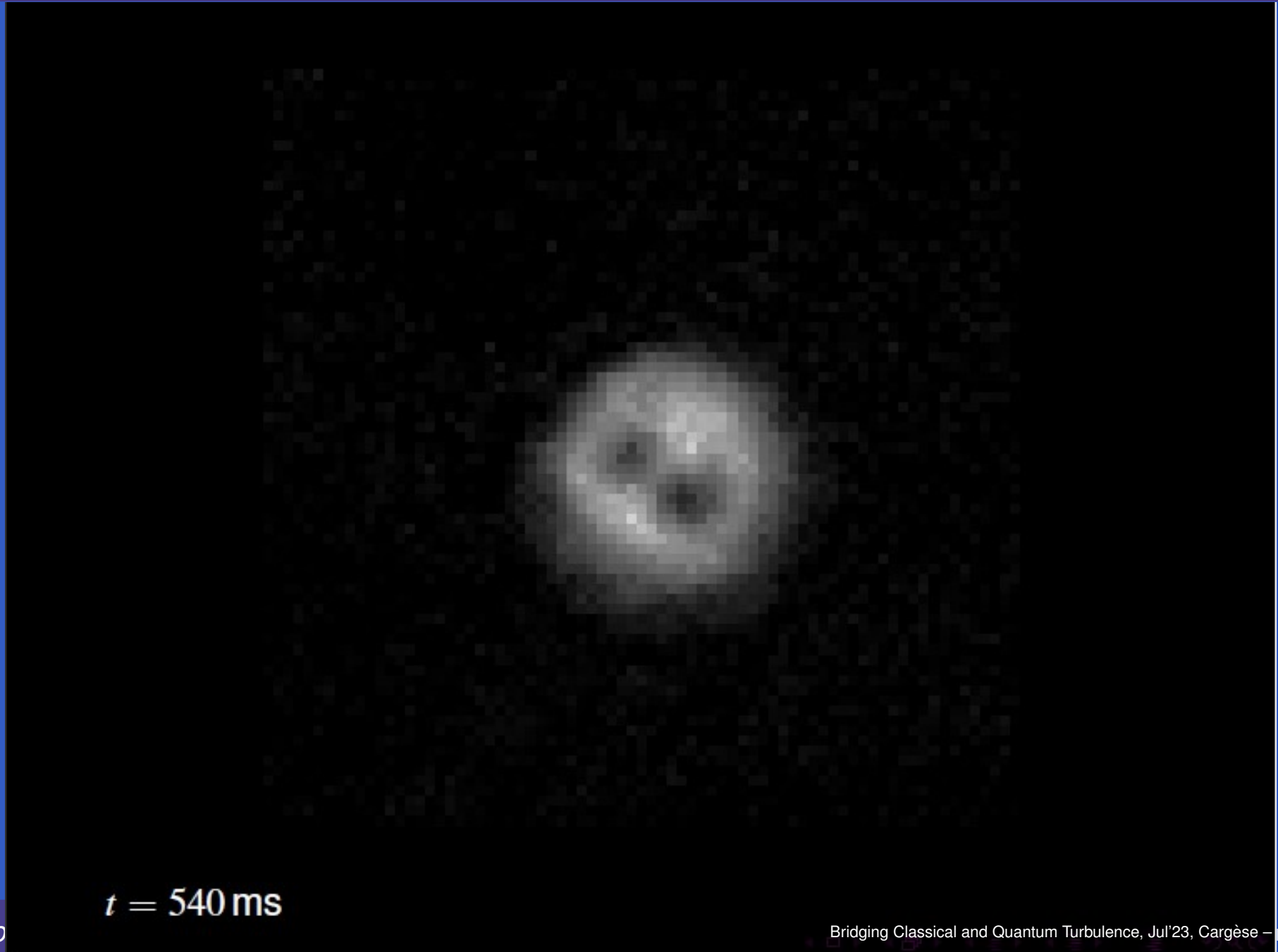


# Experimental BEC vortices: static vortex dipole



$t = 450 \text{ ms}$

# Experimental BEC vortices: static vortex dipole

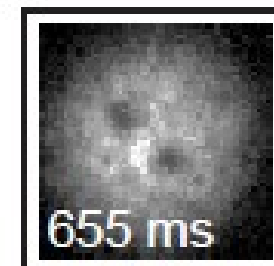
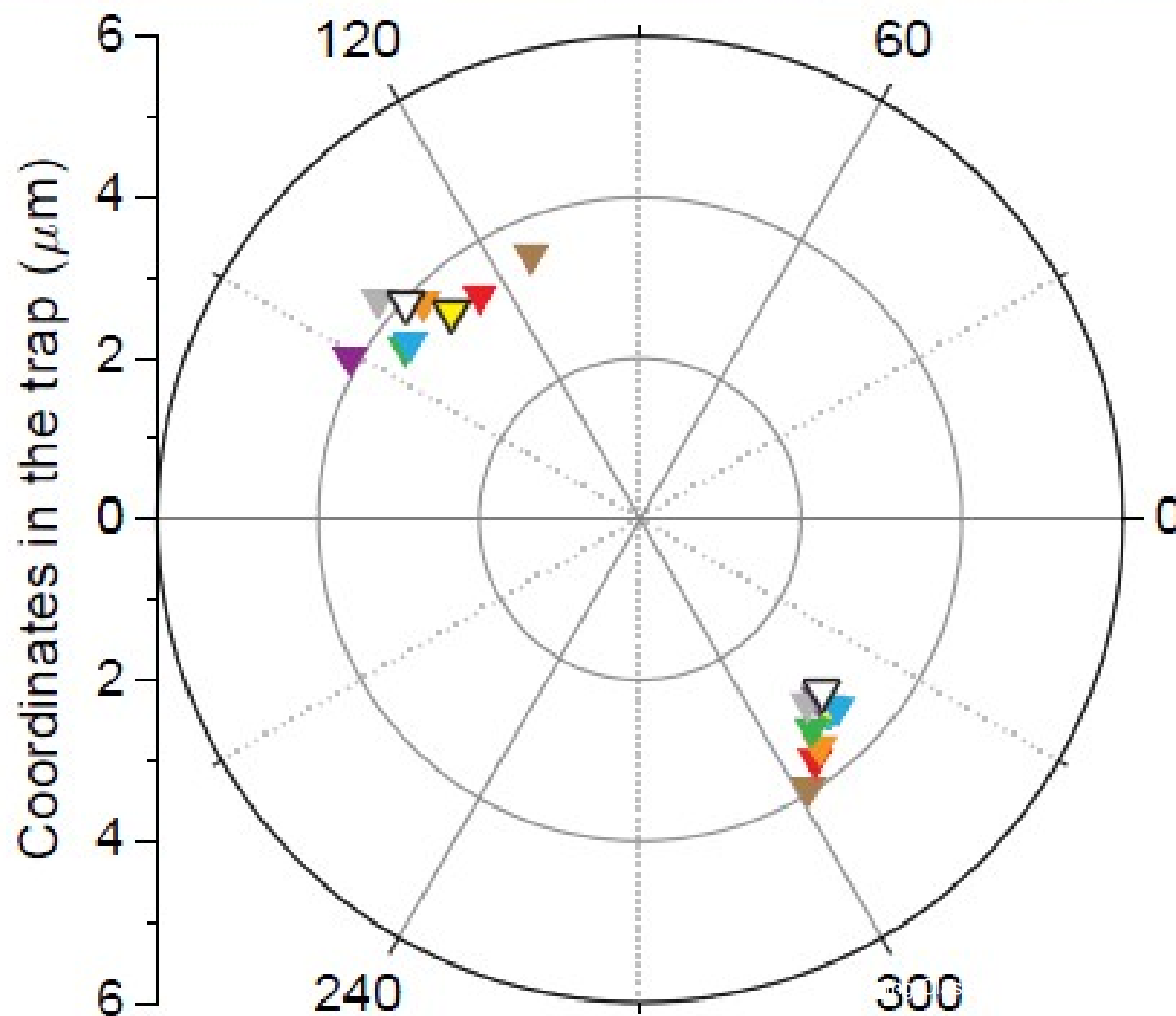
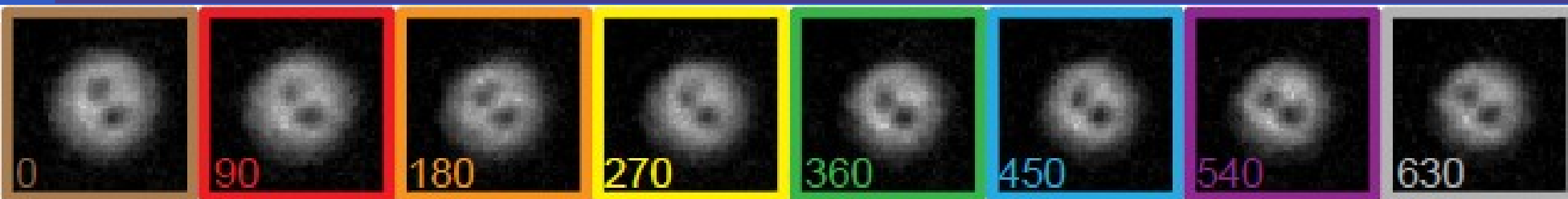


$t = 540 \text{ ms}$

# Experimental BEC vortices: static vortex dipole

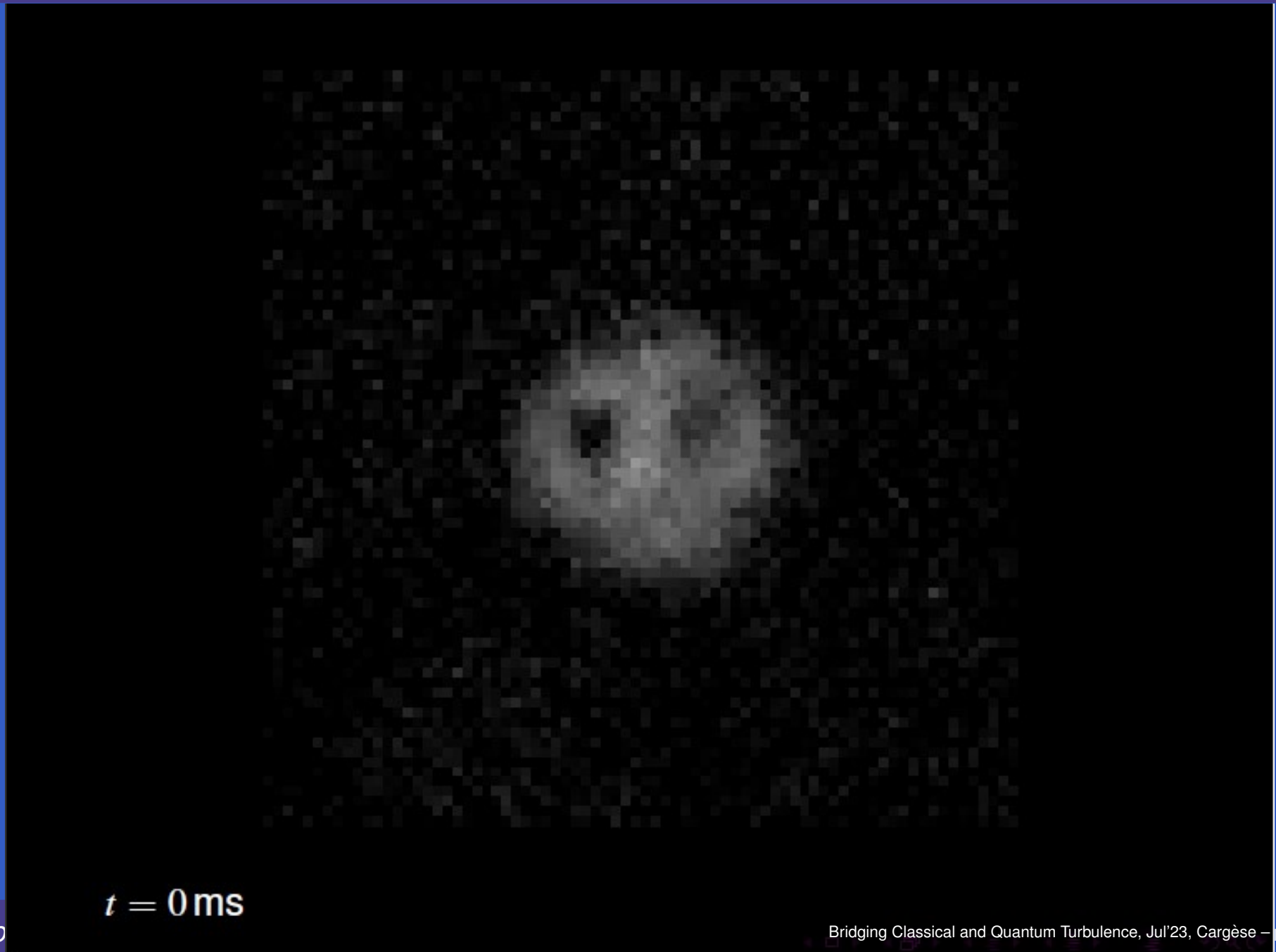


# Experimental BEC vortices: static vortex dipole



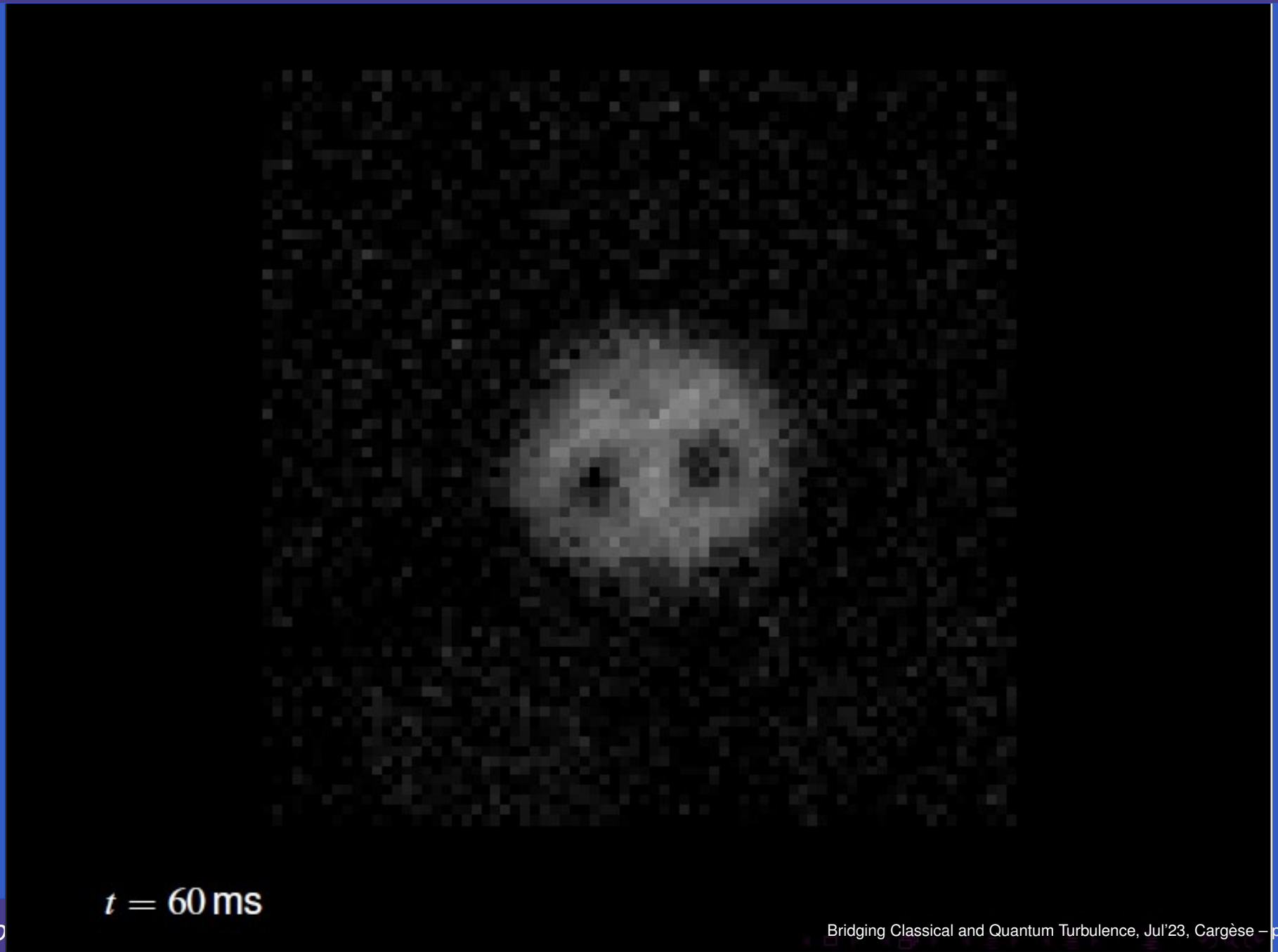
# Periodic vortex dipole

# Experimental BEC vortices: periodic cycle



$t = 0 \text{ ms}$

# Experimental BEC vortices: periodic cycle





# Experimental BEC vortices: periodic cycle



$t = 120 \text{ ms}$

# Experimental BEC vortices: periodic cycle



$t = 180 \text{ ms}$

# Experimental BEC vortices: periodic cycle



$t = 240 \text{ ms}$

# Experimental BEC vortices: periodic cycle



$t = 300$  ms

# Experimental BEC vortices: periodic cycle



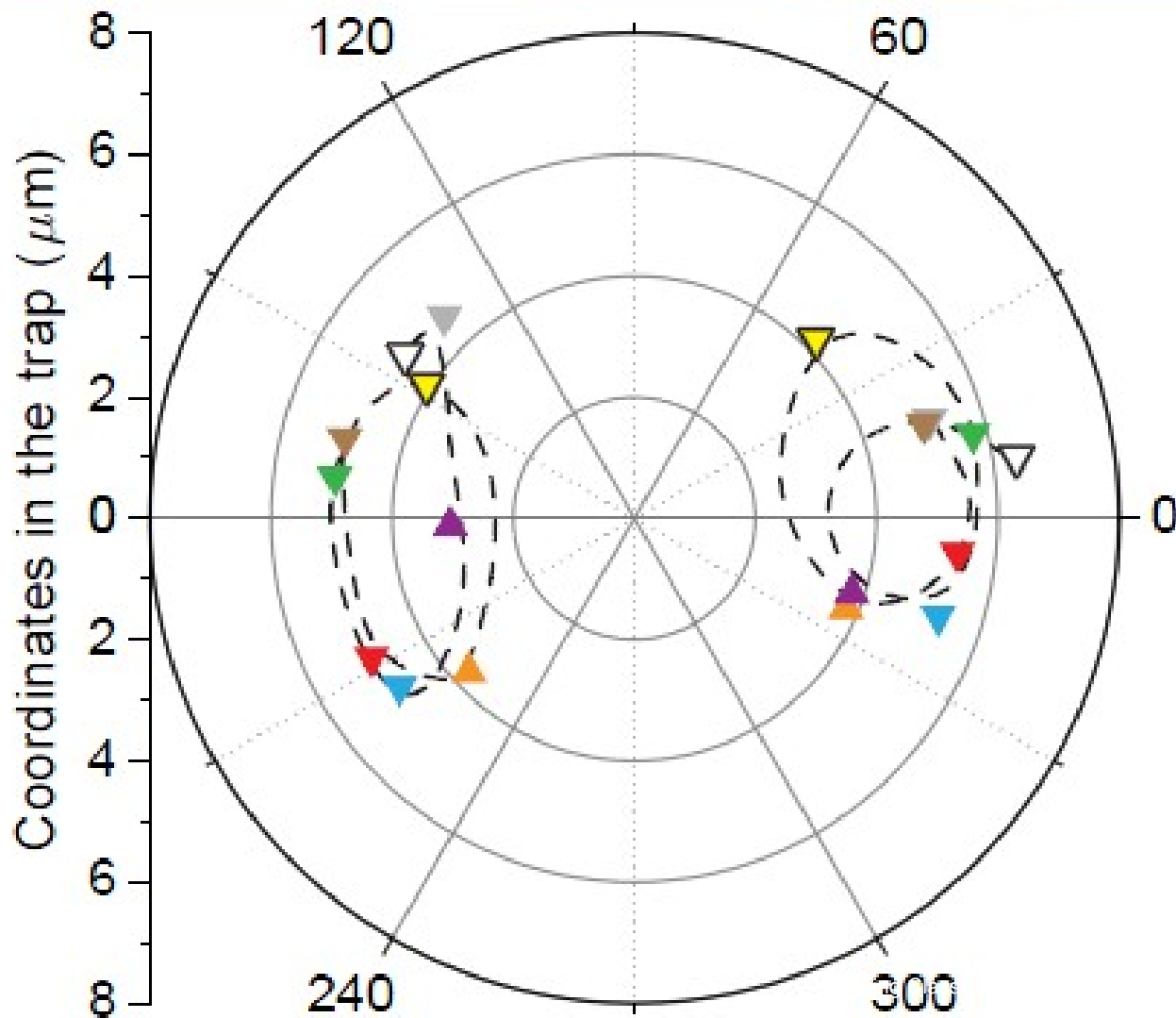
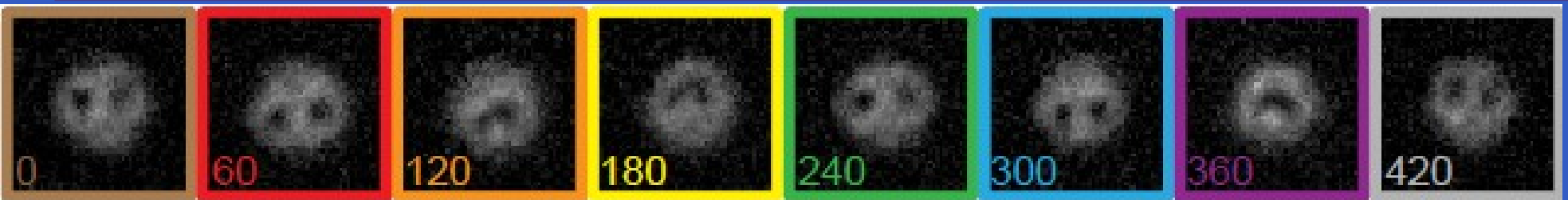
$t = 360 \text{ ms}$

# Experimental BEC vortices: periodic cycle

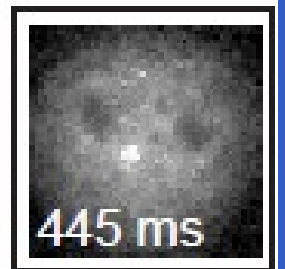


$t = 420 \text{ ms}$

# Experimental BEC vortices: periodic cycle



1%

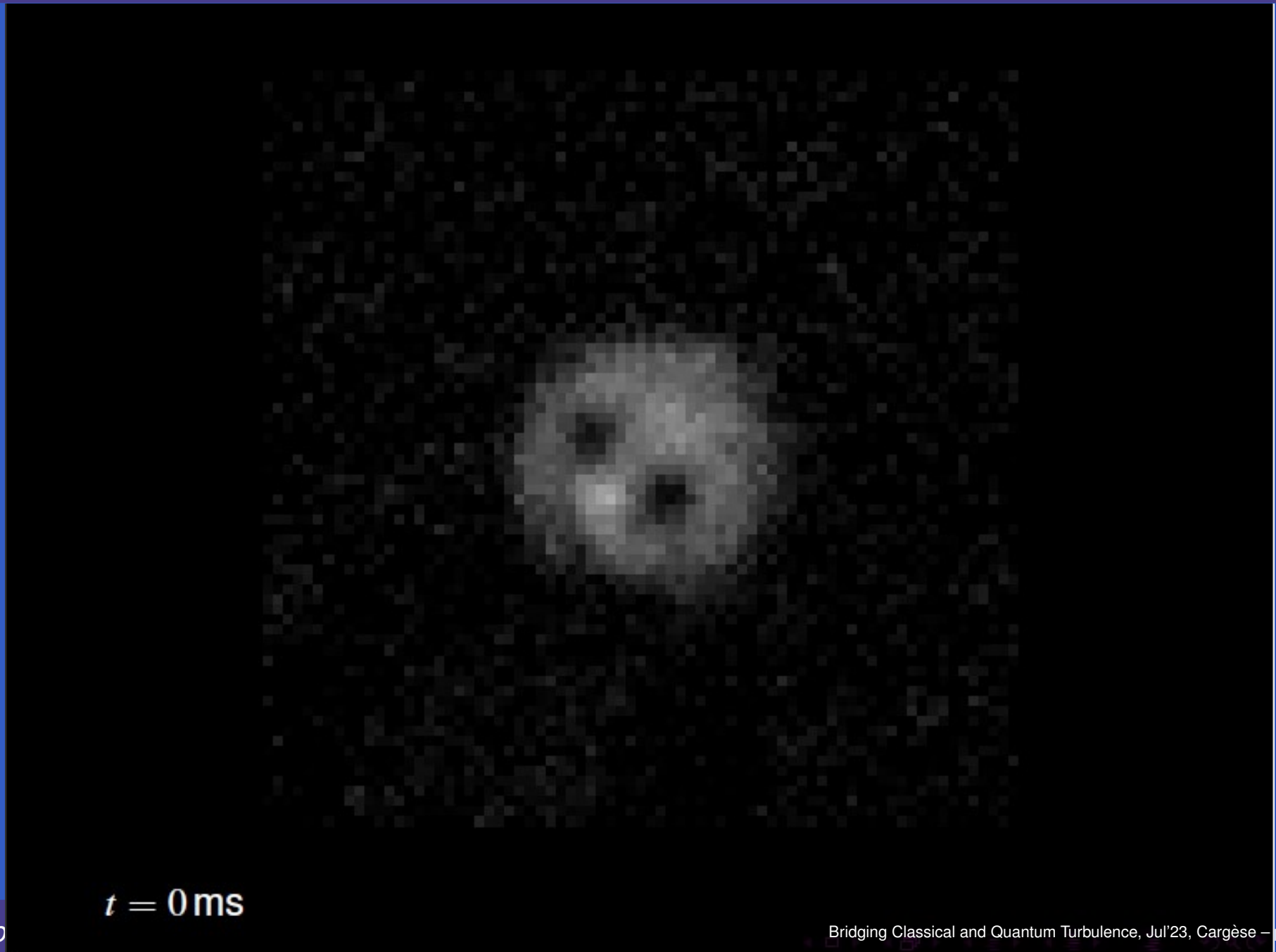


# Vortex Spirograph

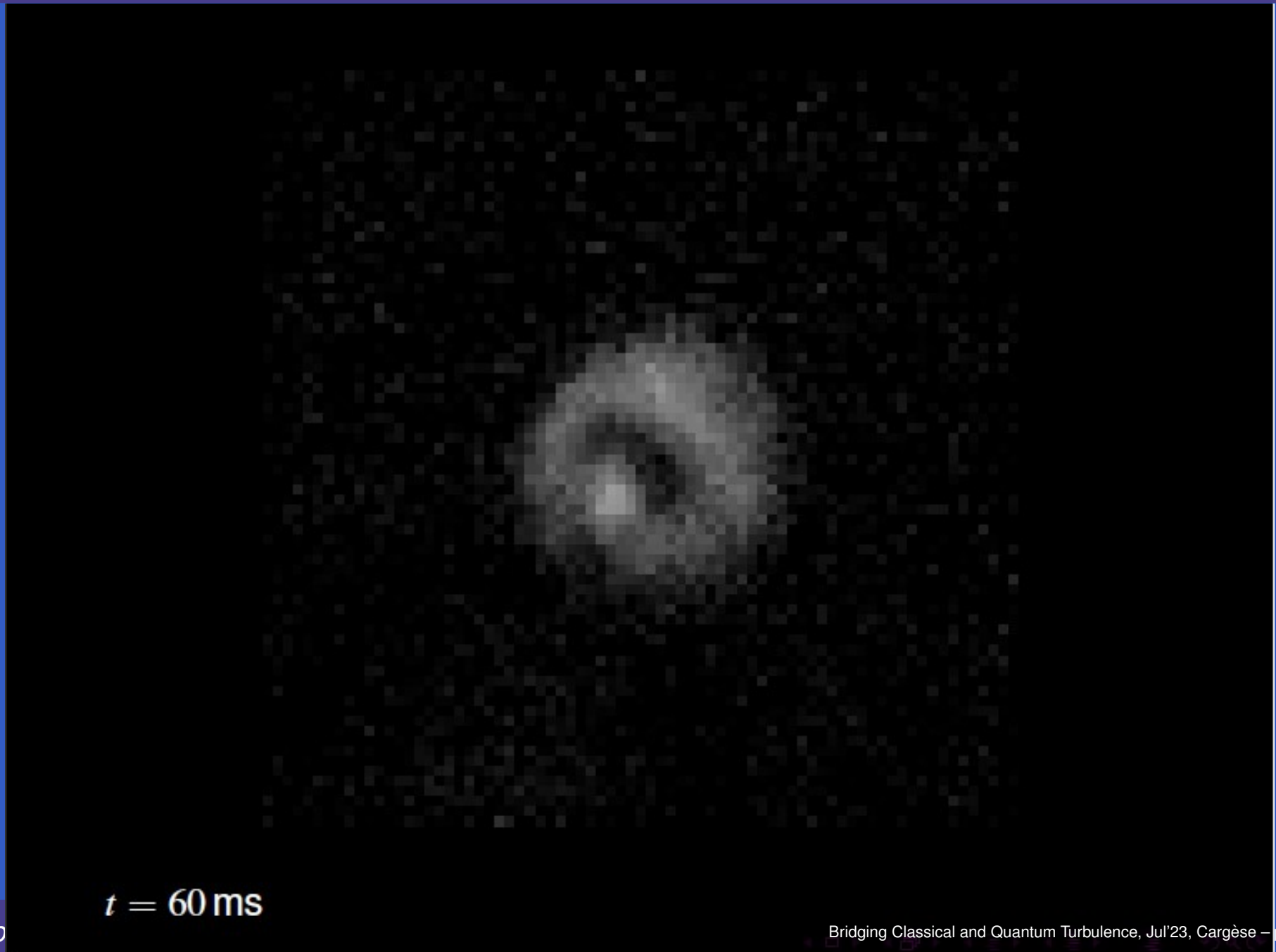




# Experimental BEC vortices: epitrochoidal motion

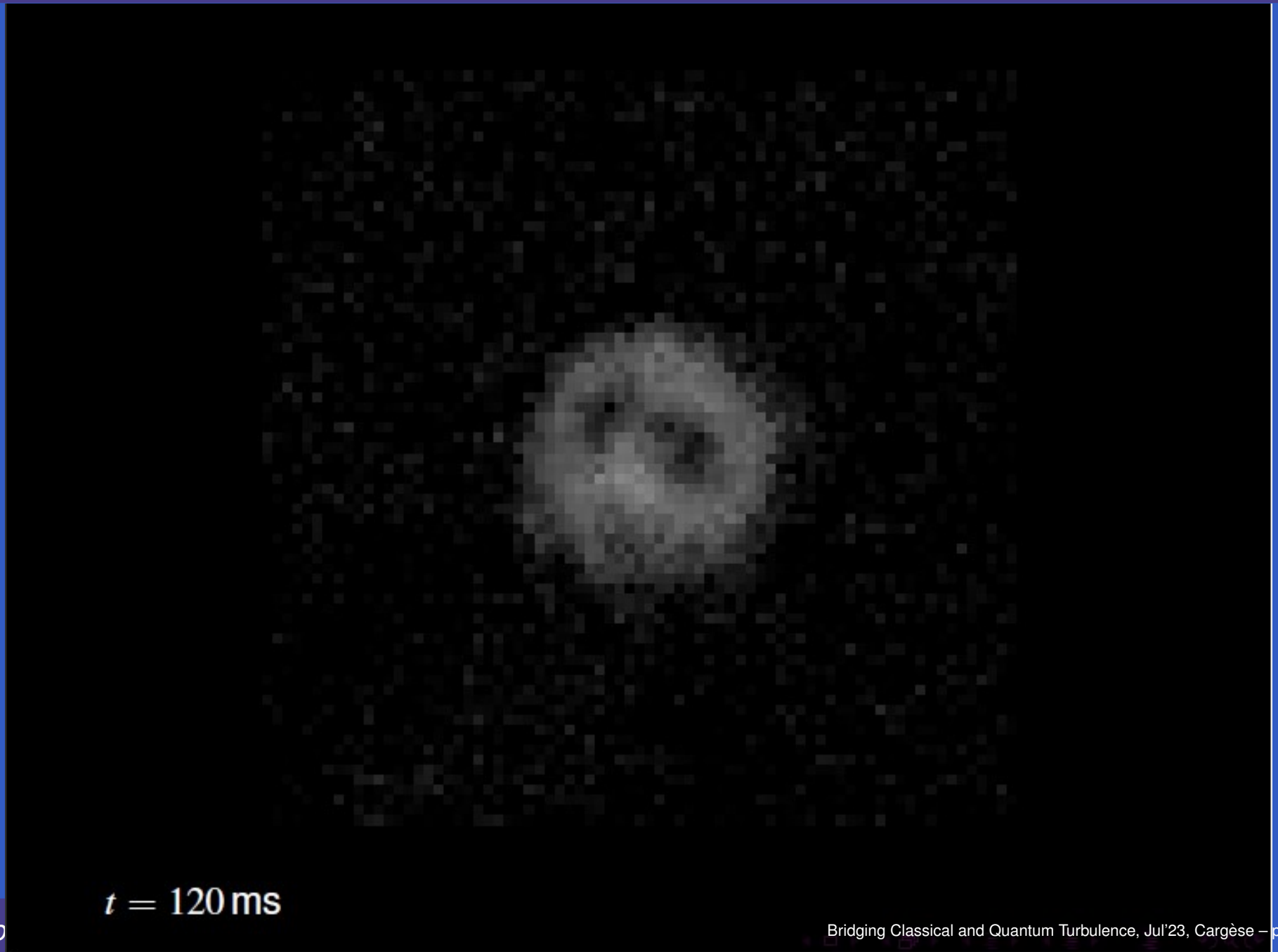


# Experimental BEC vortices: epitrochoidal motion



$t = 60 \text{ ms}$

# Experimental BEC vortices: epitrochoidal motion



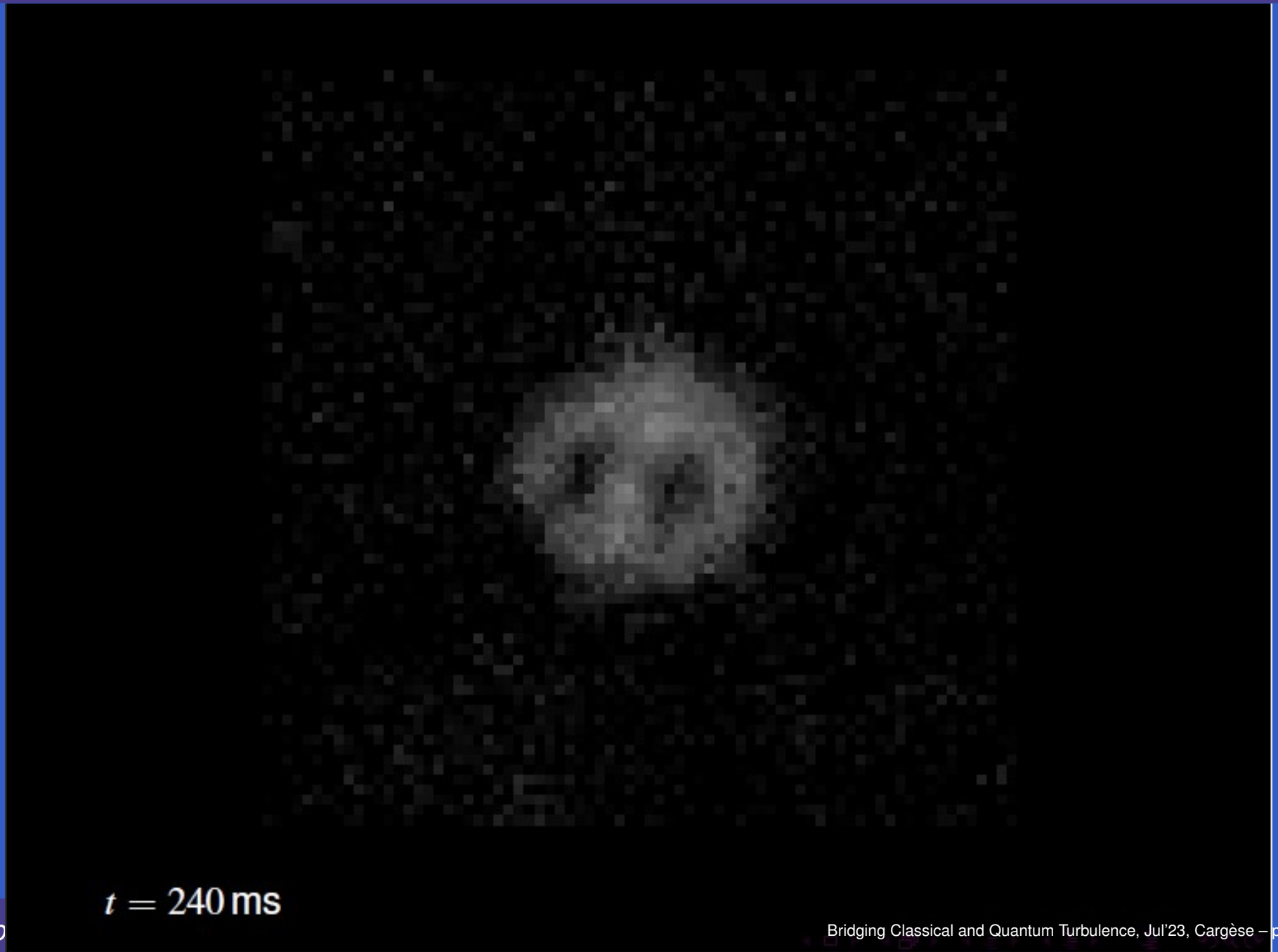
$t = 120 \text{ ms}$

# Experimental BEC vortices: epitrochoidal motion



$t = 180 \text{ ms}$

# Experimental BEC vortices: epitrochoidal motion



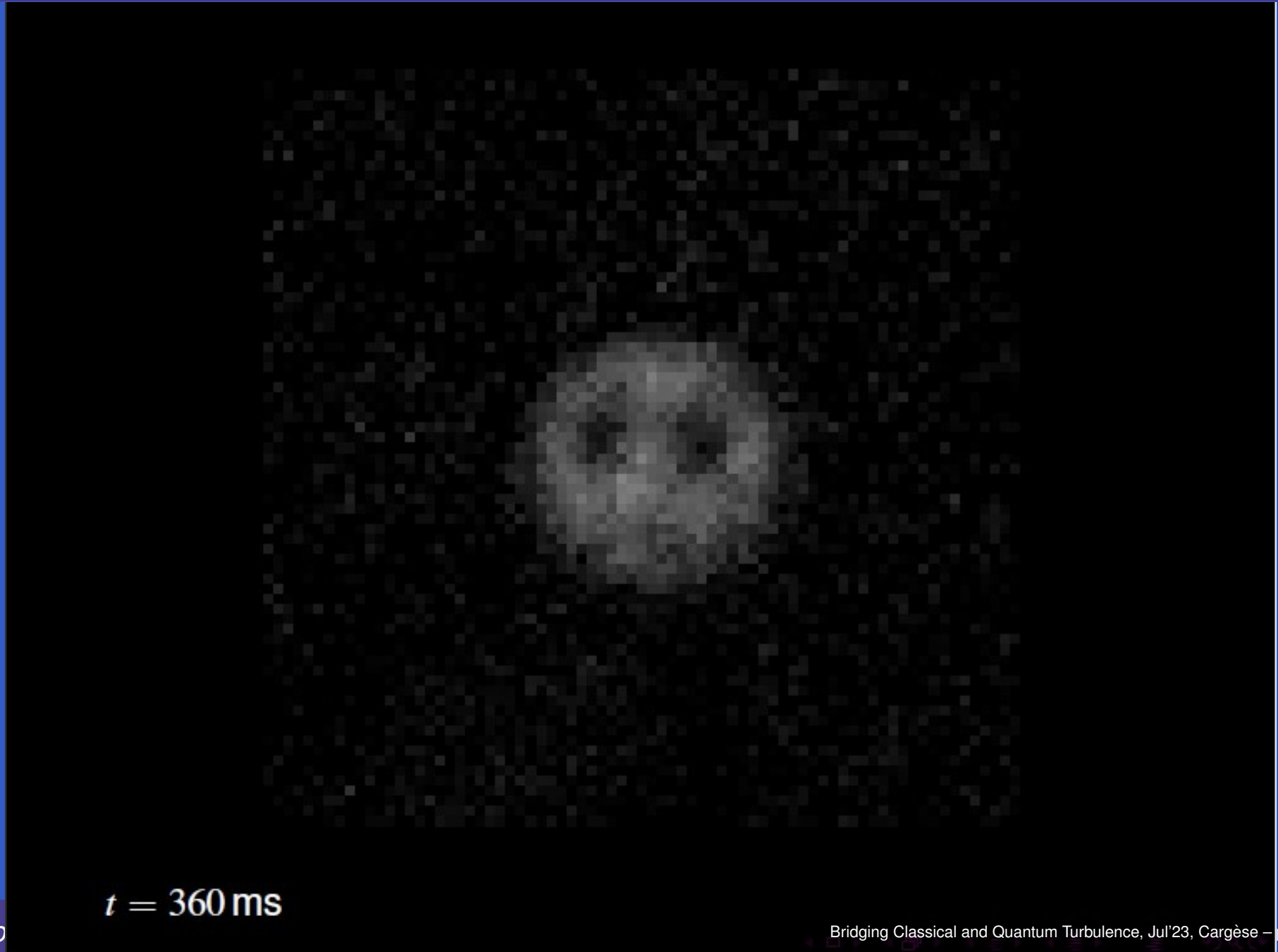
$t = 240 \text{ ms}$

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$t = 300 \text{ ms}$

# Experimental BEC vortices: epitrochoidal motion



$t = 360 \text{ ms}$

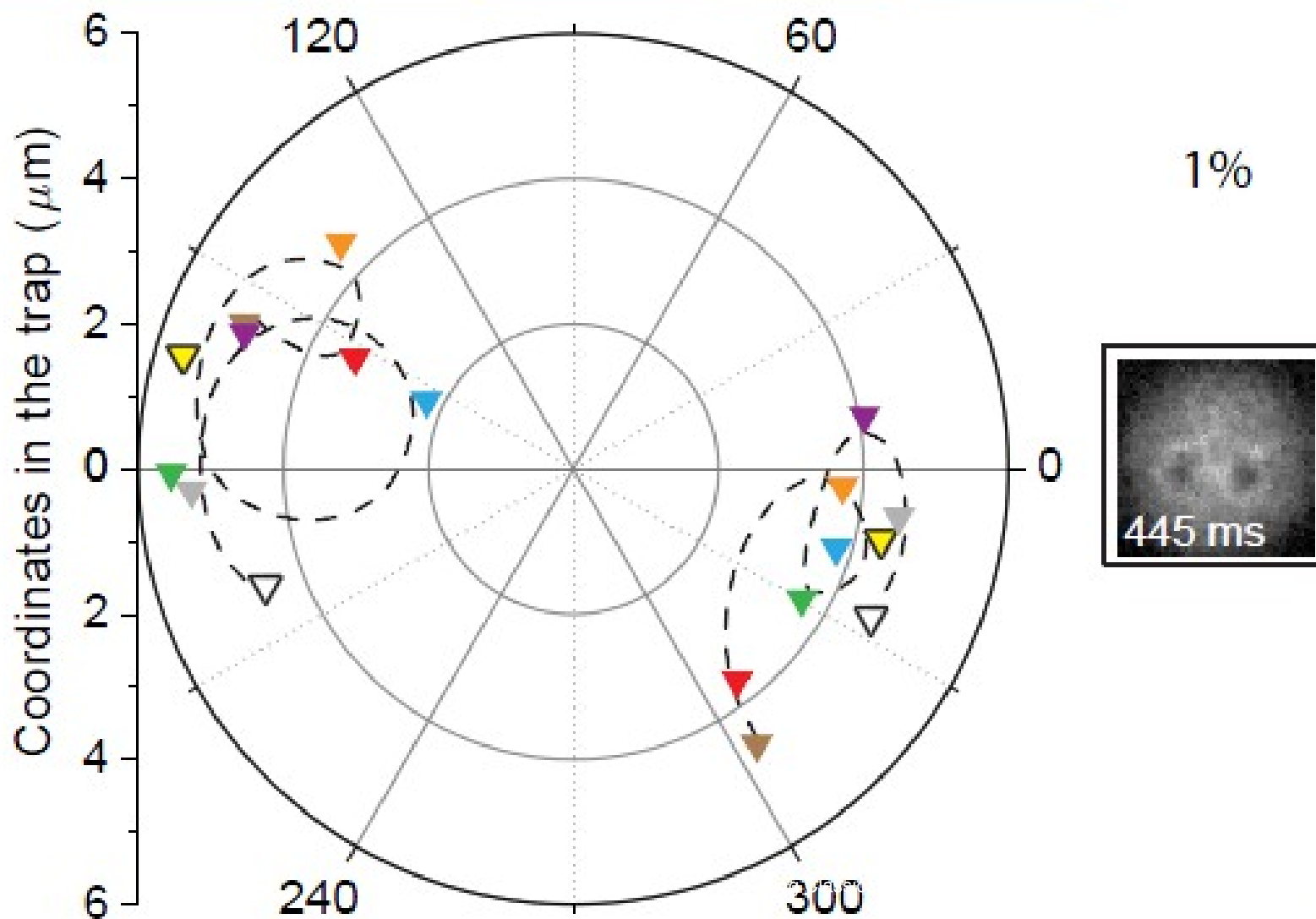
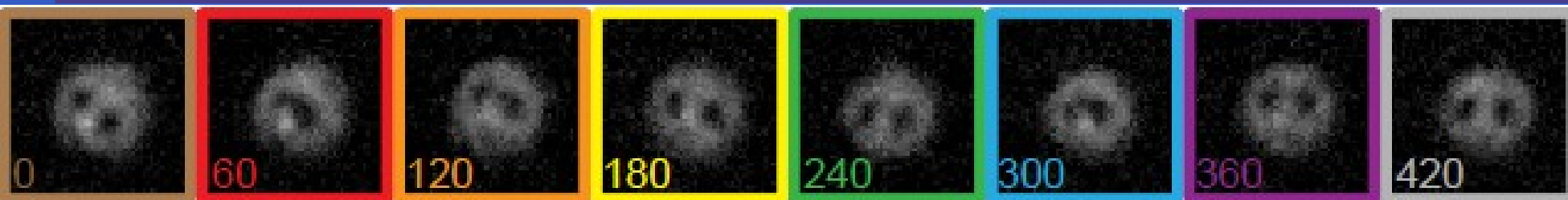
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$t = 420 \text{ ms}$



# Experimental BEC vortices: epitrochoidal motion



# Vortex dynamics and their interactions in parabolically trapped BECs

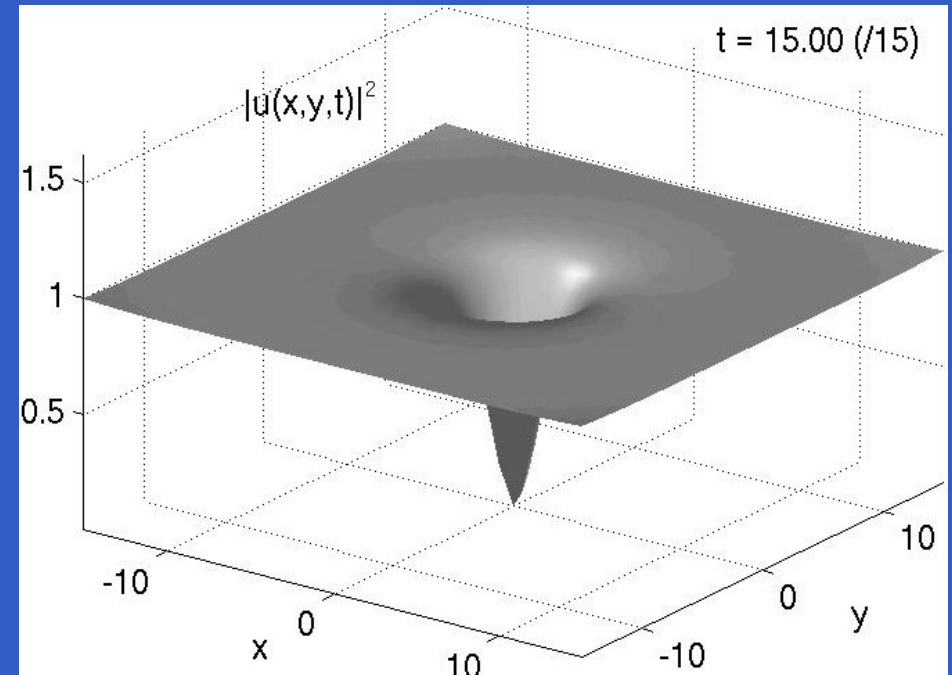
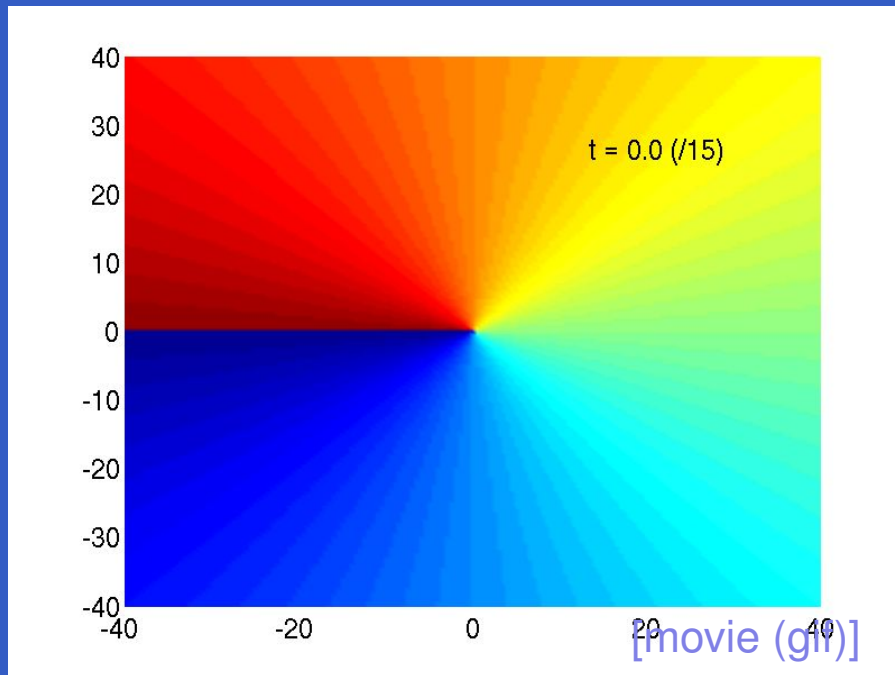
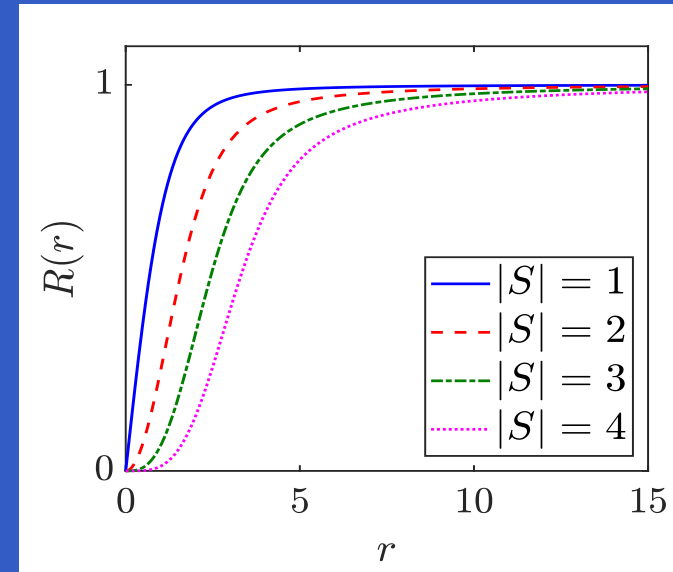
# Vortex: a single one without external potential

- Take solution with topological charge  $S$ :

$$u(x, y, t) = R(r) e^{iS\theta} e^{-i\mu t}$$

- Vortex radial profile satisfies:

$$\left( \mu - \frac{S^2}{2r^2} \right) R + \frac{1}{2r} R' + \frac{1}{2} R'' - R^3 = 0$$



# Vortex: precession in the magnetic trap (MT)

- Precession frequency for a vortex inside a MT ( $V(r) = \frac{1}{2}\Omega^2 r^2$ ) at distance  $r$  from center.

$$S \omega_{\text{pr}} = \frac{\omega_{\text{pr}}^0}{1 - \left(\frac{r}{R_{TF}}\right)^2}$$

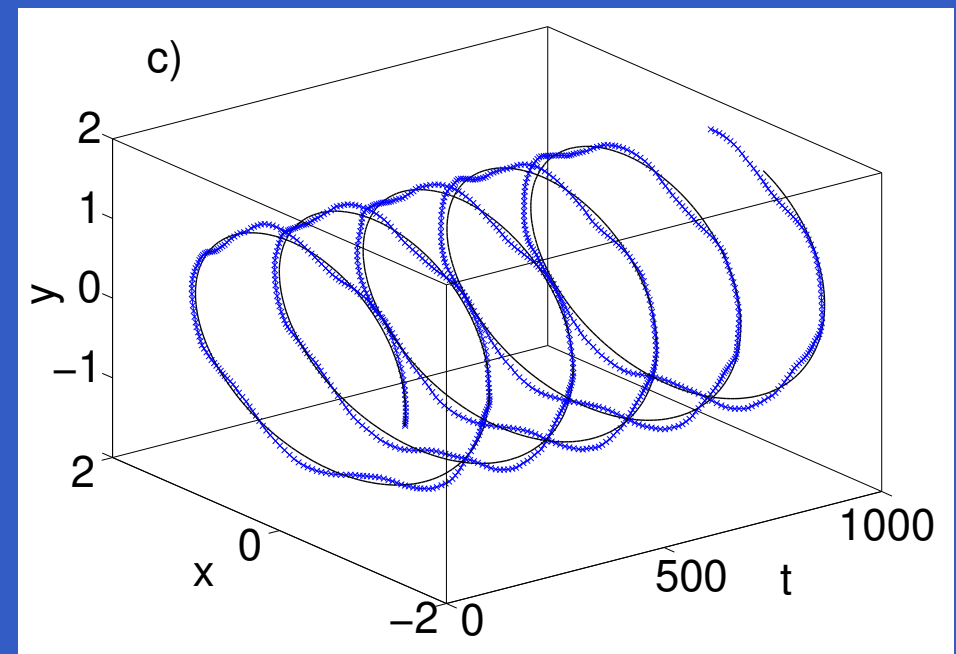
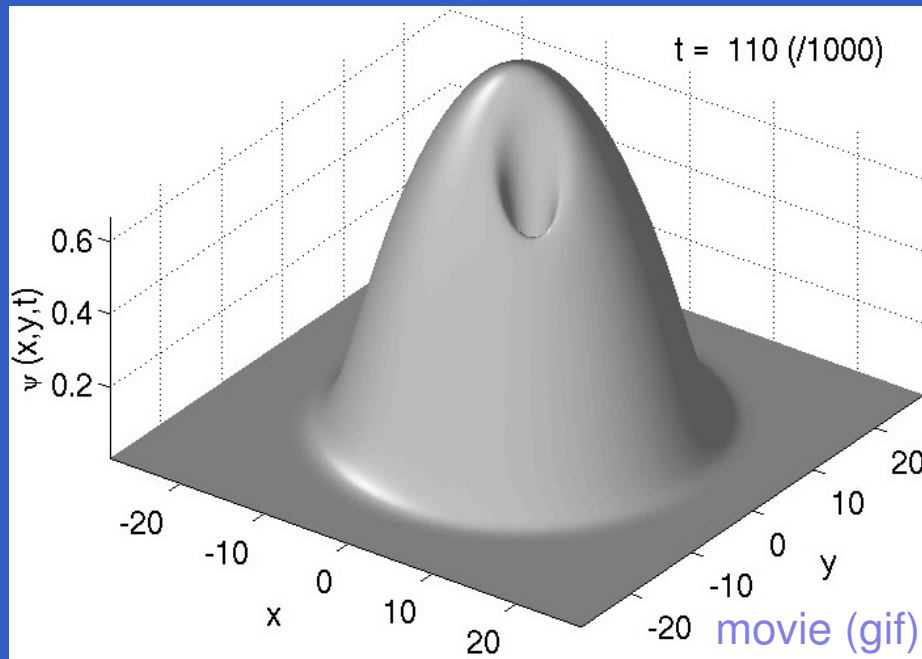
- Precession frequency close to center:  $\omega_{\text{pr}} \approx \omega_{\text{pr}}^0 \left[ = \frac{\Omega^2}{2\mu} \ln \left( A \frac{\mu}{\Omega} \right) \right]$  ( $A \approx 8.88$ ).

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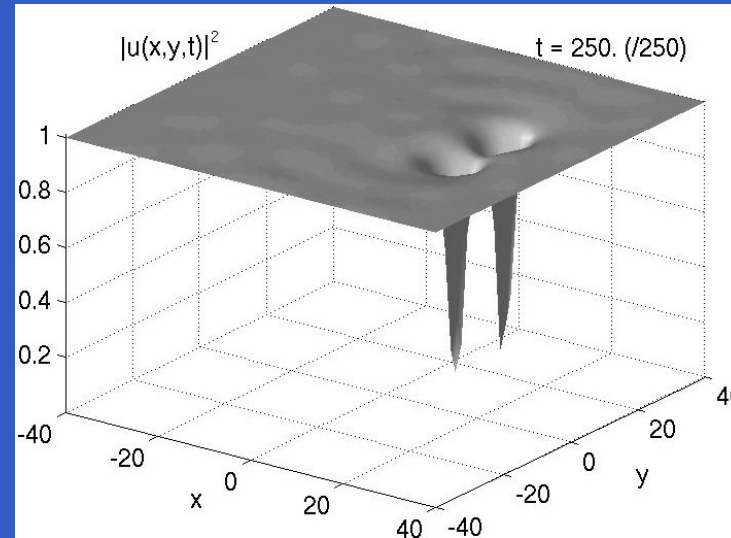
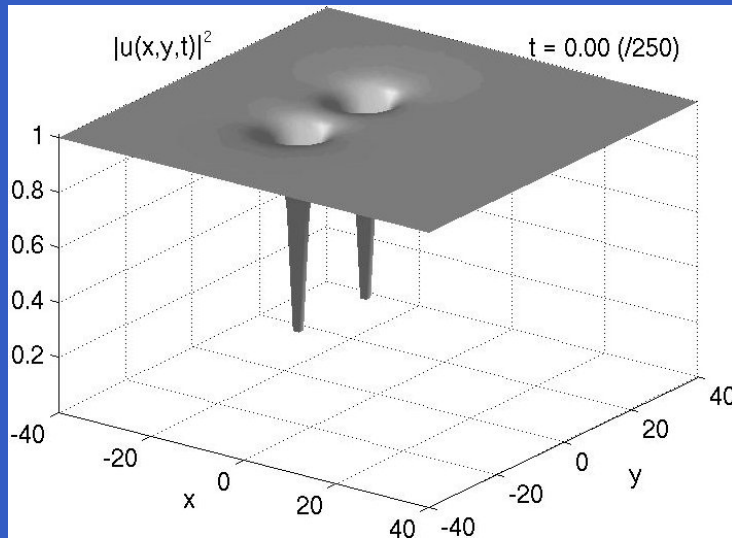
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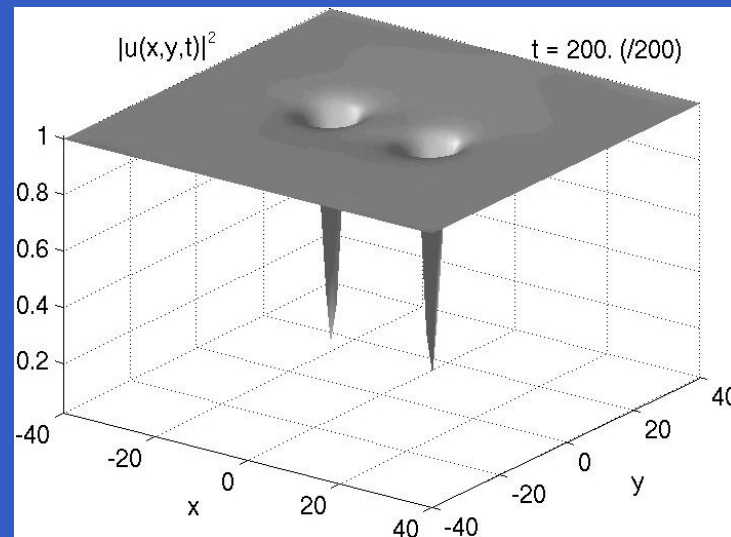
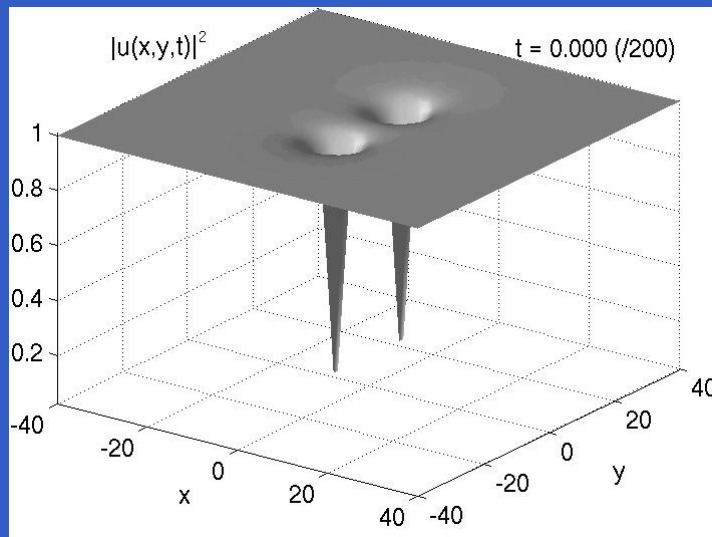
# Vortex interactions: pairwise dynamics

- Opposite charge (vortex dipole)  $\rightarrow$  run parallel to each other



[movie]

- Same charge (vortex pair)  $\rightarrow$  rotate about mid-point



[movie]

# Vortices: vortex-vortex interactions

- Movement induced by phase gradient and density gradient [Matched asymptotics  $\rightarrow$  Kivshar+Pismen+...].
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$$\dot{x}_1 = -\omega_{\text{vort}} S_2 \frac{y_1 - y_2}{2r_{12}^2},$$
$$\dot{y}_1 = +\omega_{\text{vort}} S_2 \frac{x_1 - x_2}{2r_{12}^2},$$

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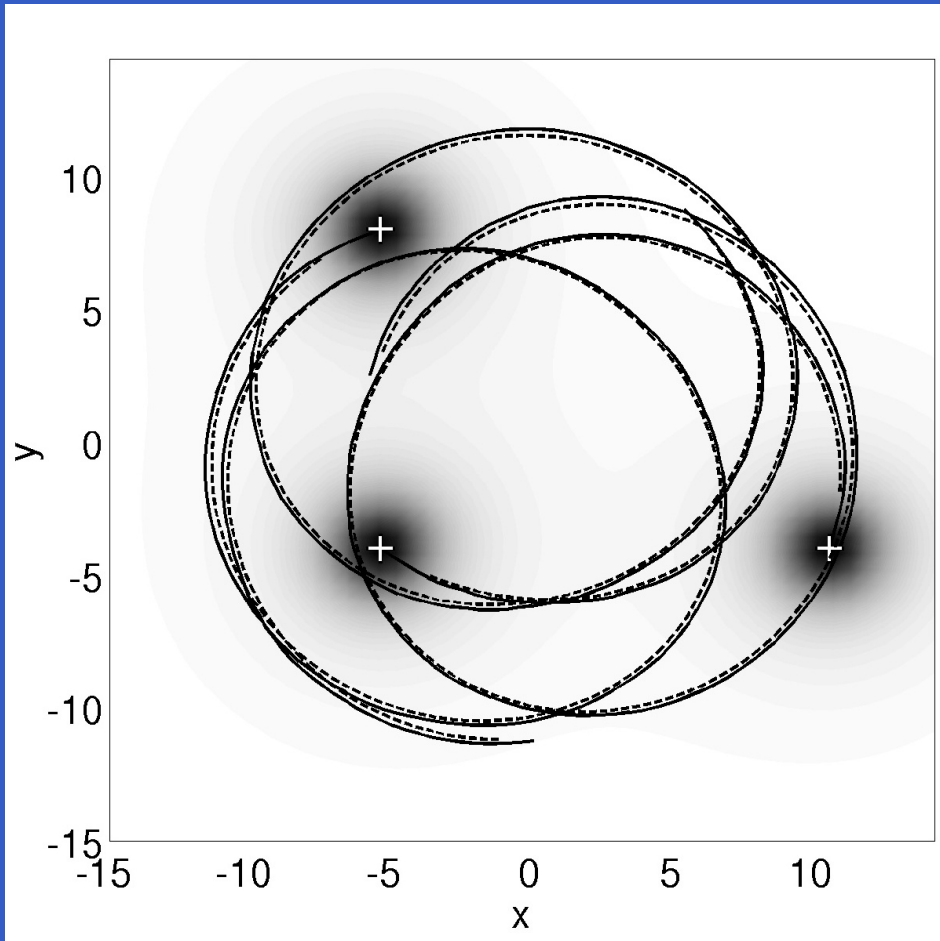
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- Superposition of  $N$  vortices:

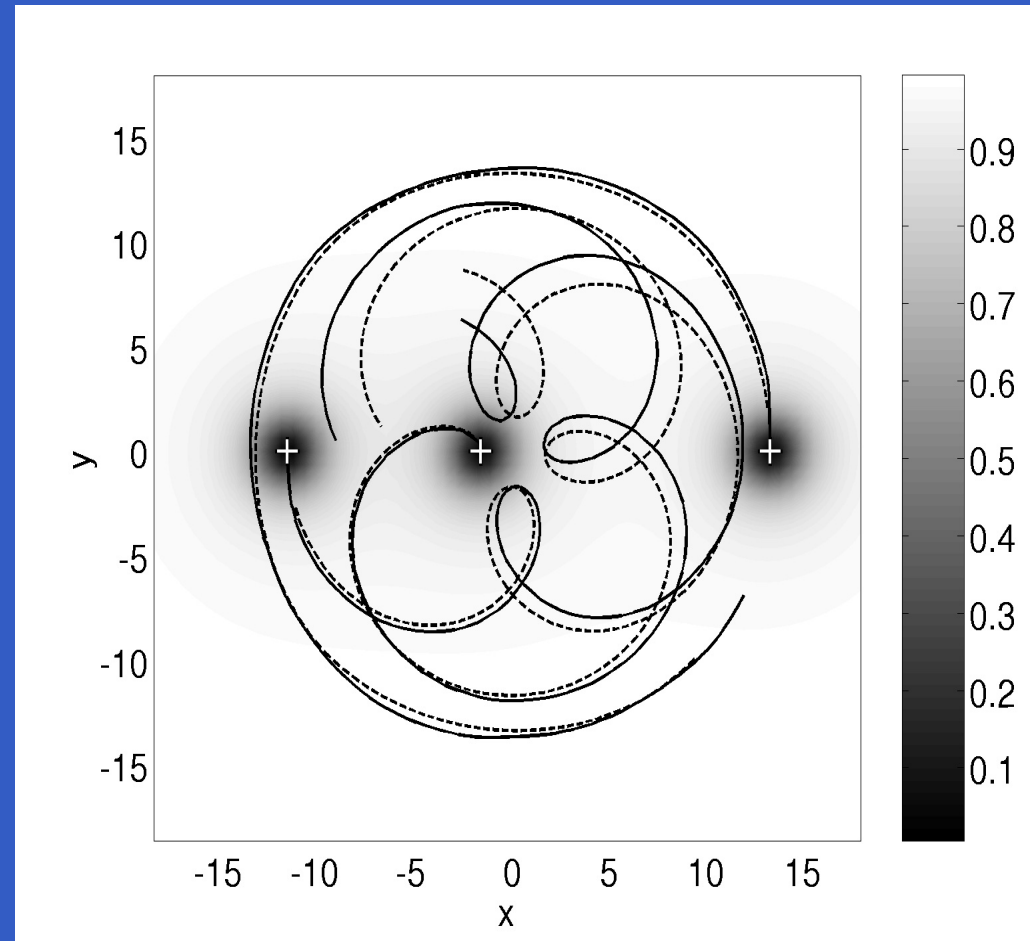
$$\dot{x}_m = -\frac{\omega_{\text{vort}}}{2} \sum_{n \neq m}^N S_m \frac{y_m - y_n}{r_{mn}^2}$$

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# Vortex trajectories (no MT): PDE vs ODE



[movie]



[movie]

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$$\dot{y}_m = S_m \frac{\omega_{\text{pr}}^0}{1 - r_m^2} x_m + \frac{\omega_{\text{vort}}}{2} \sum_{n \neq m}^N S_n \frac{x_m - x_n}{r_{mn}^2}$$

- Conserved quantities: Hamiltonian and angular momentum:

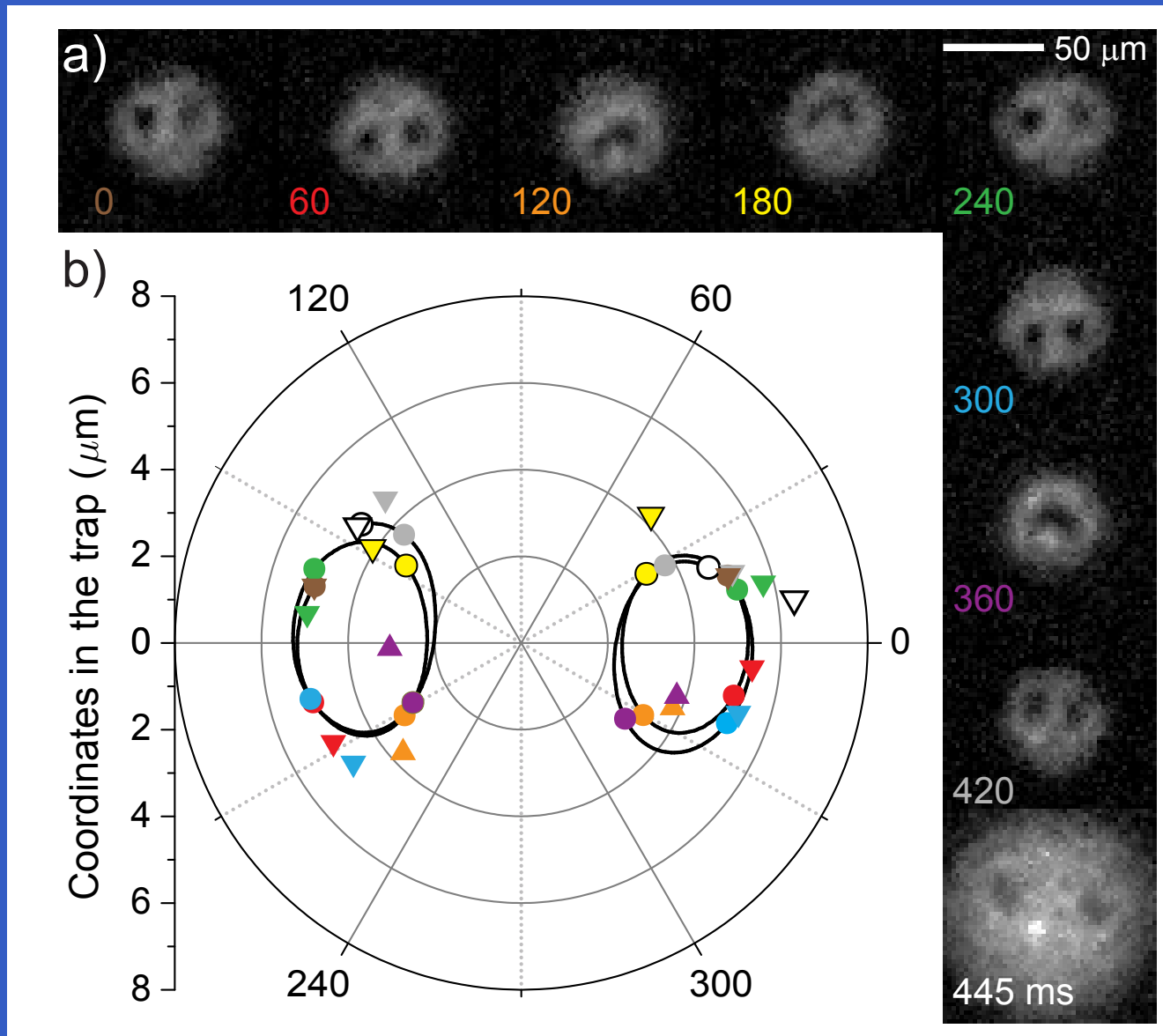
$$H = -\frac{\omega_{\text{pr}}^0}{2} \sum_{n=1}^N \ln(1 - r_n^2) + \frac{\omega_{\text{vort}}}{4} \sum_{n=1}^N \sum_{m \neq n}^N S_m S_n \ln(r_{mn}^2),$$

$$L_0 = \sum_{n=1}^N S_n r_n^2.$$

Vortex dipoles inside MT:  
OPPOSITE charge pair:

$$S_1 = 1 \text{ \& } S_2 = -1$$

# Experiments (David Hall) vs. theory





# Stationary (non-rotating) equilibria:

- Equilibrium for diametrically opposed (symmetric) vortices:

$$r_{\text{eq}} = 2 \sqrt{\frac{\omega_{\text{vort}}}{4\omega_{\text{pr}}^0 + \omega_{\text{vort}}}}$$

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rotations with frequency:

$$\omega_{\text{eq}} = \sqrt{2} \omega_{\text{pr}}^0 \left( 1 + \frac{\omega_{\text{vort}}}{4\omega_{\text{pr}}^0} \right)^{3/2}$$

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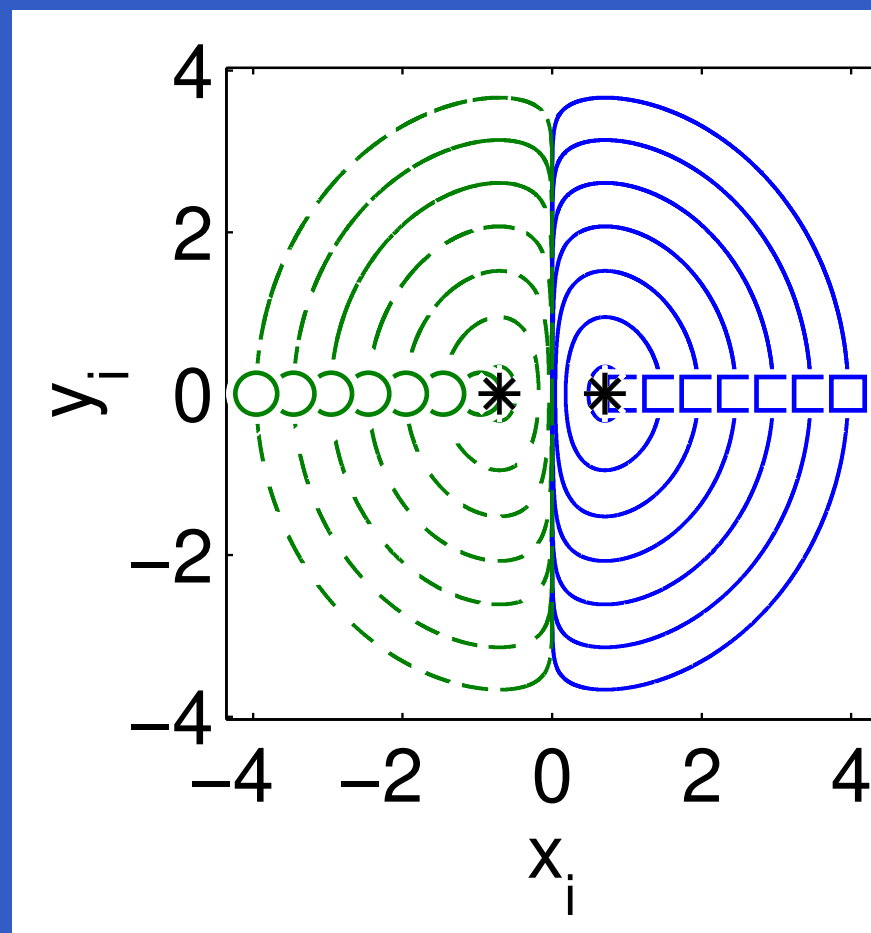
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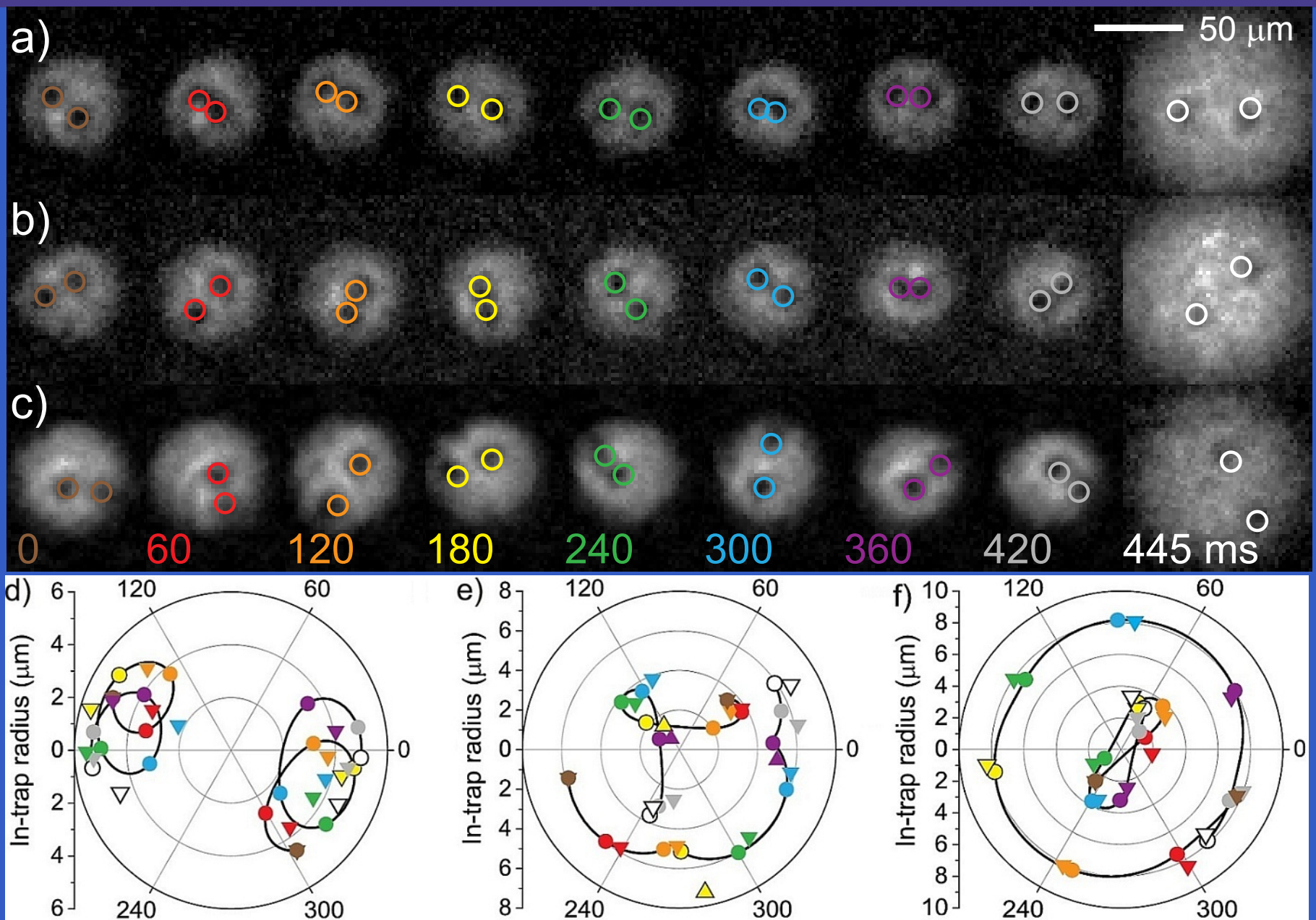
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- Rotation frequency around the equilibrium  $\omega_{\text{eq}} > 0 \Rightarrow$  always STABLE !  
[contrast with the same-charge vortex case]



# More experiments (David Hall) → the Spirograph



# Asymmetric co-rotating equilibria:

- Consider diametrically opposed vortices but asymmetric wrt to the center:  $z_1 = r_1 \exp(i\omega_{\text{orb}}t)$  and  $z_2 = -r_2 \exp(i\omega_{\text{orb}}t)$  with  $r_1 > r_2$ .
- Asymmetric co-rotating equilibrium positions:

$$\omega_{\text{pr}}^0 \left( \frac{1}{1 - r_1^2} + \frac{1}{1 - r_2^2} \right) - \frac{\omega_{\text{vort}}}{2r_1 r_2} = 0.$$

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- Rotating with freq:

$$\omega_{\text{orb}} = \frac{1}{2} \left[ \omega_{\text{pr}}^0 (\alpha - \beta) + \gamma \omega_{\text{vort}} \left( \frac{r_1}{r_2} - \frac{r_2}{r_1} \right) \right],$$

where

$$\alpha = \frac{1}{1-r_1^2}, \quad \beta = \frac{1}{1-r_2^2}, \quad \text{and} \quad \gamma = \frac{1}{2r_1^2r_2^2}.$$

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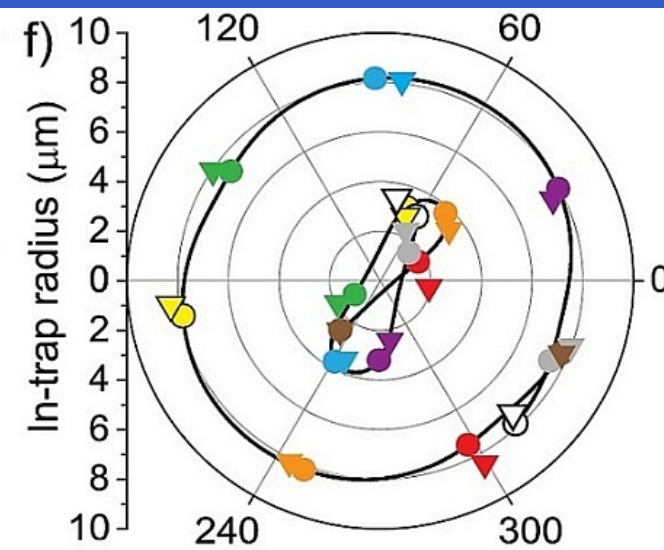
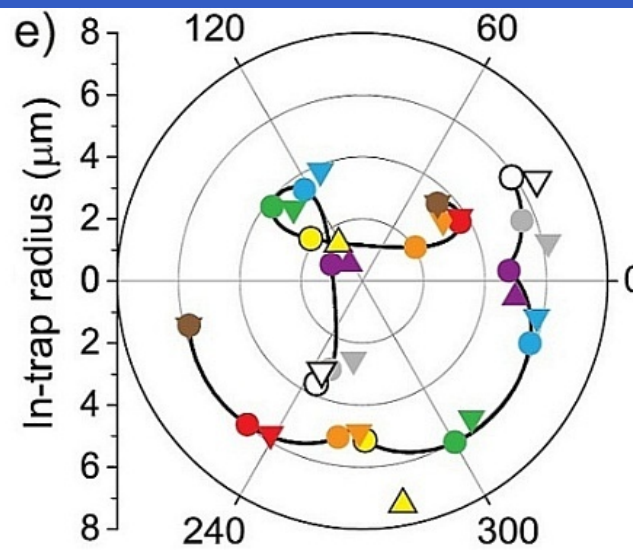
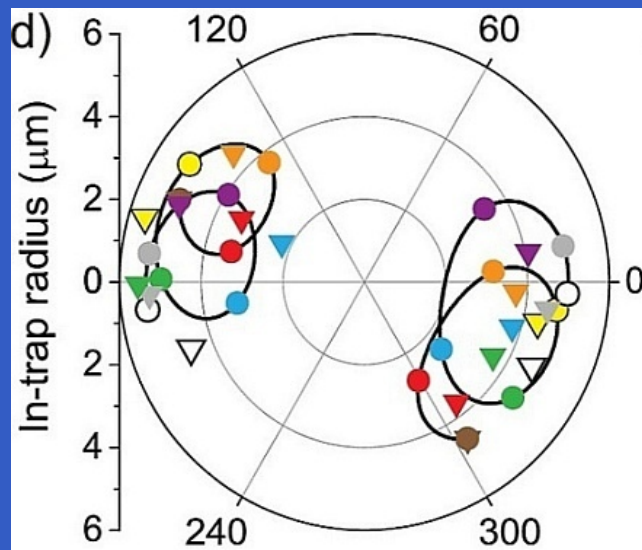
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- In co-rot frame: perturbs about equilibria result in rotating orbits.
  - epitrochoidal motion (Spirograph) in the original frame !!!
  - Generic motion: **quasi-periodic epitrochoids**.

# Motion about asymmetric equilibria: epitrochoids!

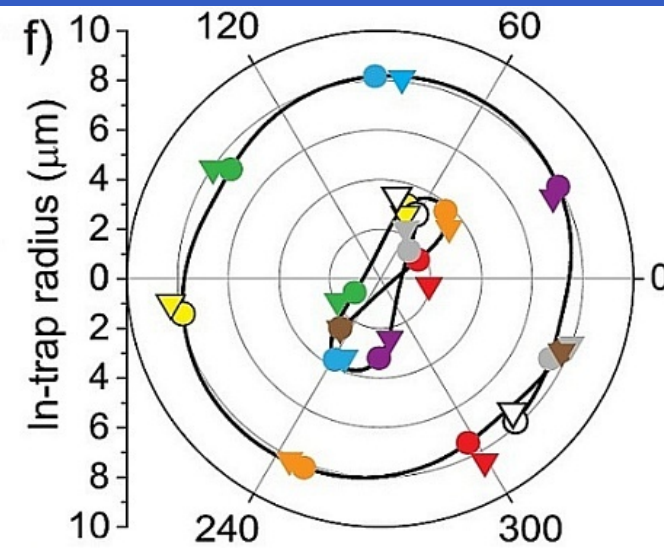
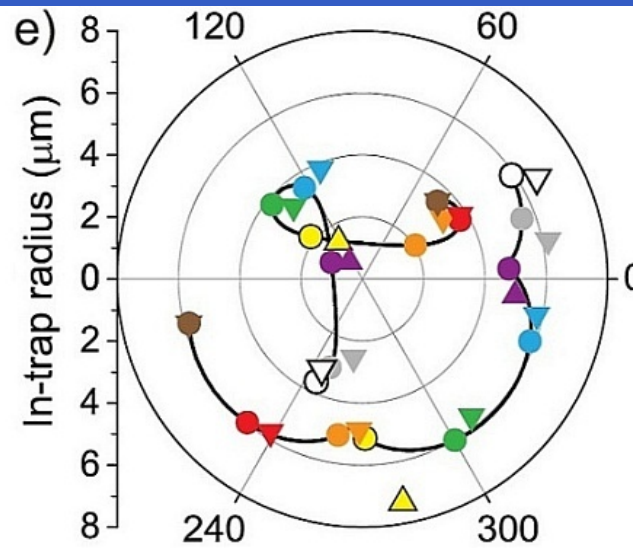
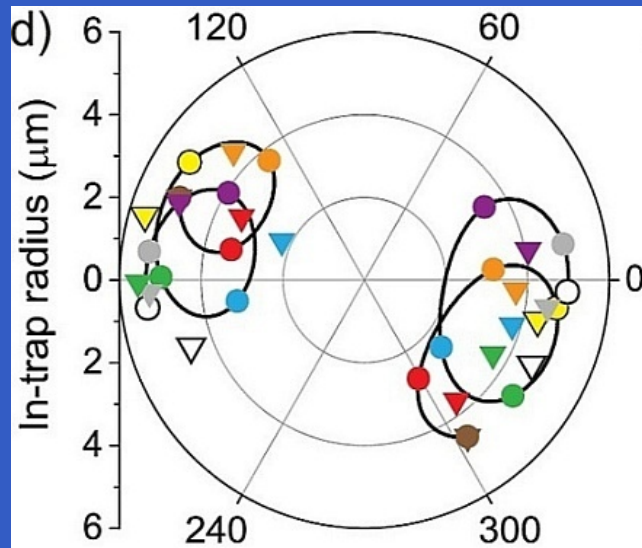
● On the original (lab.) frame. Experiments:



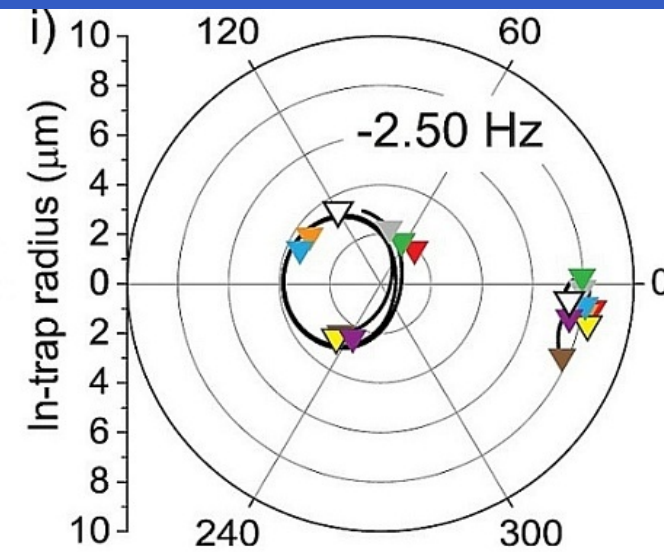
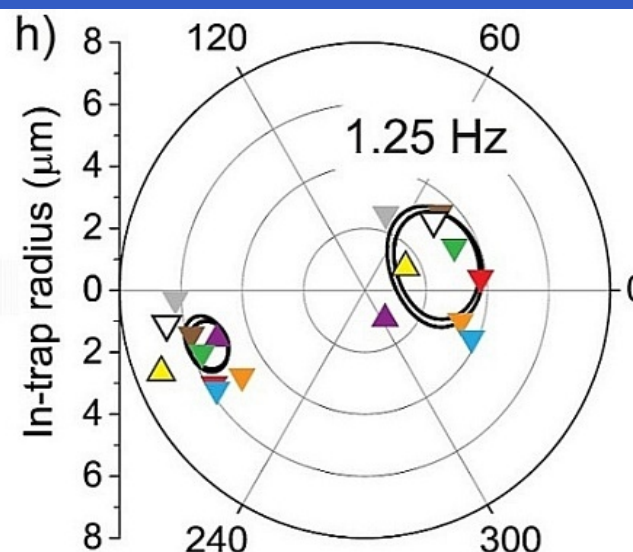
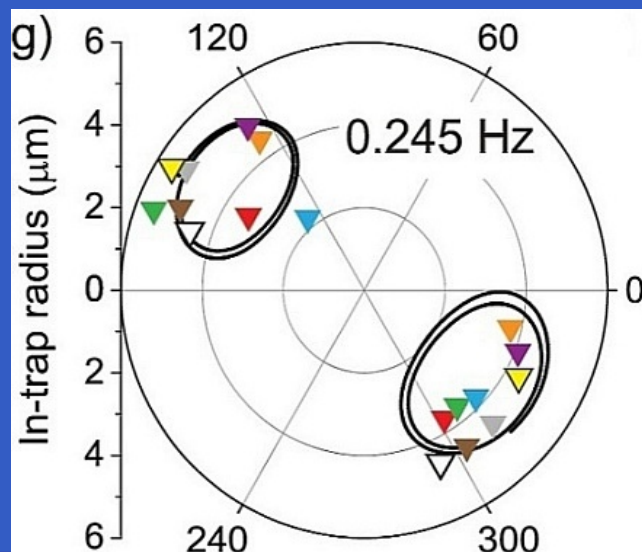


# Motion about asymmetric equilibria: epitrochoids!

● On the original (lab.) frame. Experiments:



● On the co-rotating ( $\omega_{\text{orb}}$ ) reference frame:



Vortex pairs inside MT:  
SAME charge pair:

$$S_1 = 1 = S_2$$

# Adim. and transform to co-rotating polar coord.:

- Adimensionalize:  $(X, Y) = \frac{(x, y)}{R_{\text{TF}}}$ ,  $\tau = \omega_{\text{pr}}^0 t$ ,  $c \equiv \frac{1}{2} \frac{\omega_{\text{vort}}}{\omega_{\text{pr}}^0}$ .

- Transf. co-rot to polar:  $X_n = r_n \cos \theta_n$ ,  $Y_n = r_n \sin \theta_n$ ,  $\delta_{mn} = \theta_m - \theta_n$

$$\dot{r}_m = -\frac{c r_n \sin \delta_{mn}}{r_{mn}},$$

$$\dot{\delta}_{mn} = (r_m^2 - r_n^2) \frac{c \cos \delta_{mn}}{r_m r_n r_{mn}^2} + \frac{1}{1 - r_m^2} - \frac{1}{1 - r_n^2}.$$

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- Steady state is  $r_1 = r_2 = r_*$  (for ANY  $r_*$ ) and  $\theta_1 - \theta_2 = \pi$ , in co-rot:

$$\omega_{\text{orb}} \equiv \dot{\theta}_1 = \dot{\theta}_2 = \frac{c}{2r_*^2} + \frac{1}{1 - r_*^2}.$$

$[\omega_{\text{orb}} > 0 \Rightarrow \nexists$  no-rotating equilibria  $\rightarrow$  only ROTATING equilibria]

# Epitrochoids about SYMMETRIC rotating equilibria:

● Perturbations about equilibria:  $r_m = r_* + R_m$  and  $\delta_{mn} = \pi + \Delta_m$

● Eqs on the pert.:

$$\ddot{R}_m = -\omega_{\text{ep}}^2 (R_n - R_m),$$

$$\ddot{\Delta}_m = -\omega_{\text{ep}}^2 (\Delta_m - \Delta_n),$$

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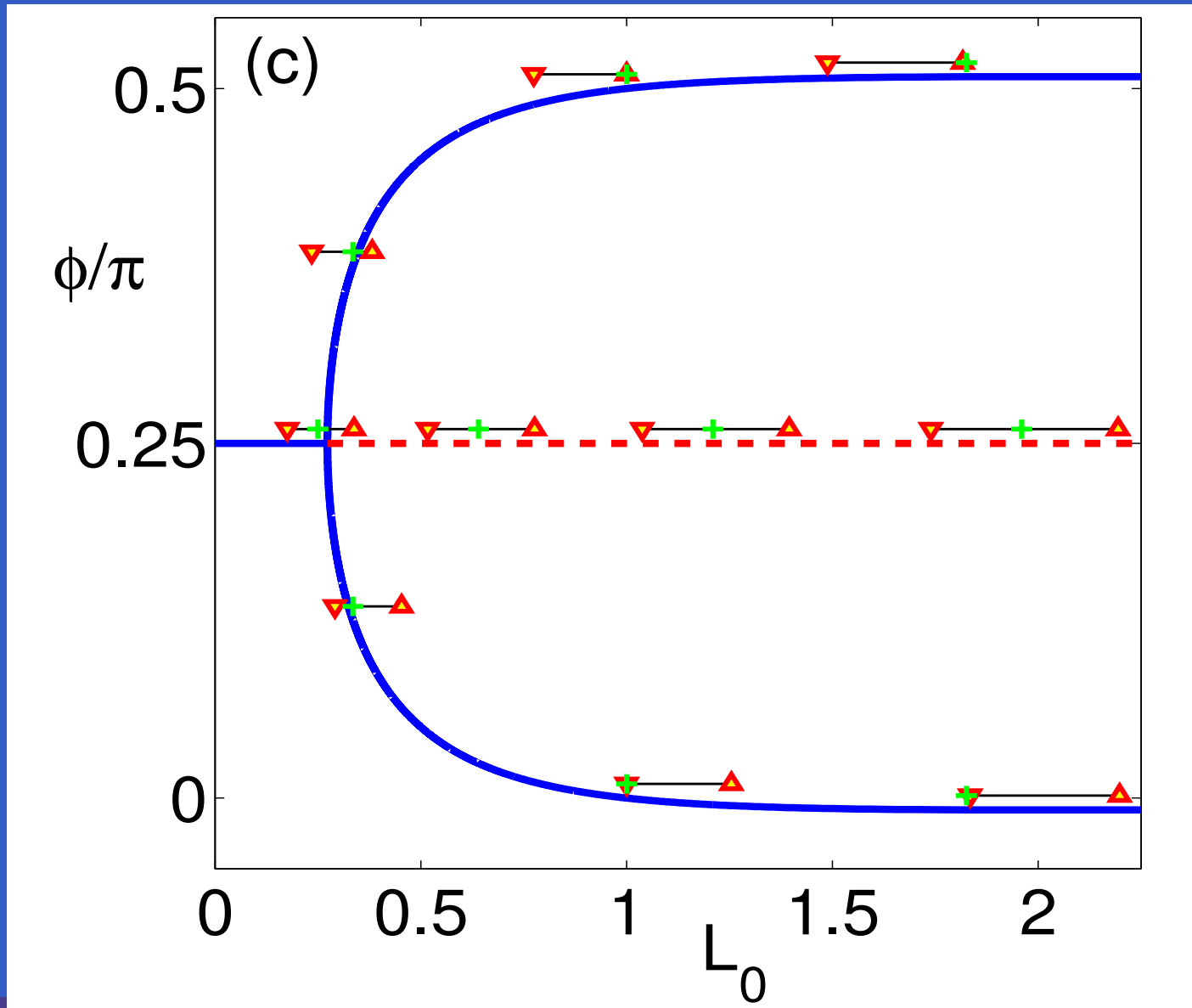
• If  $r_* > r_{\text{crit}} \Rightarrow$  **INSTABILITY !!!**

$\rightarrow$  What does it imply for dynamics ?

$\rightarrow$  **Q:** Can this be observed in experiment ???

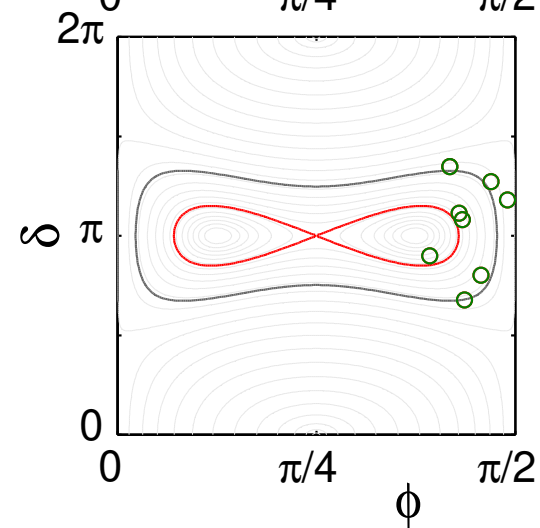
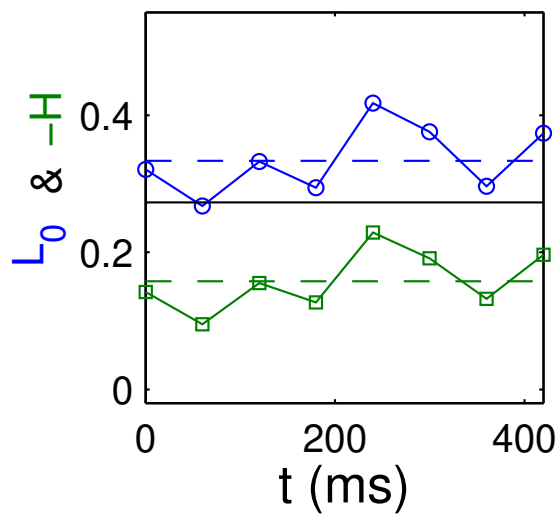
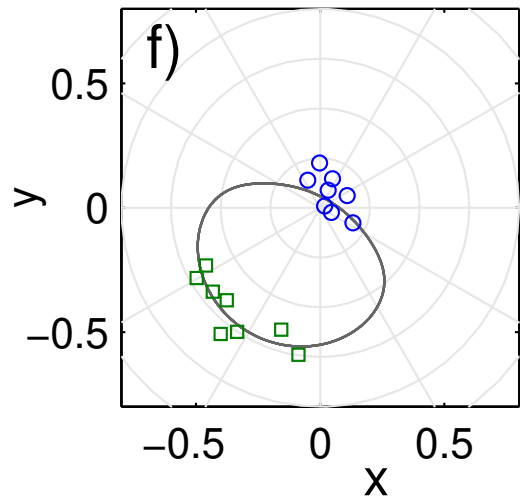
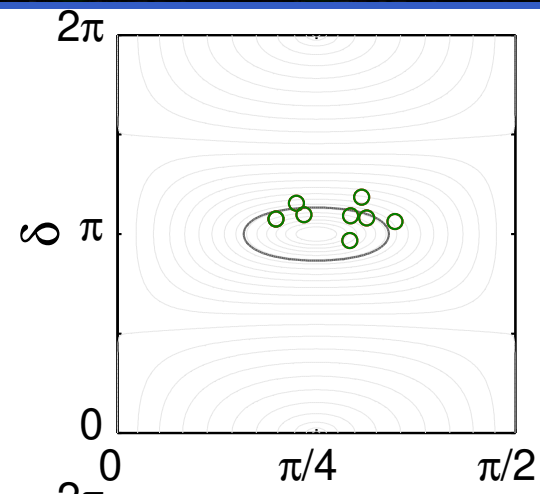
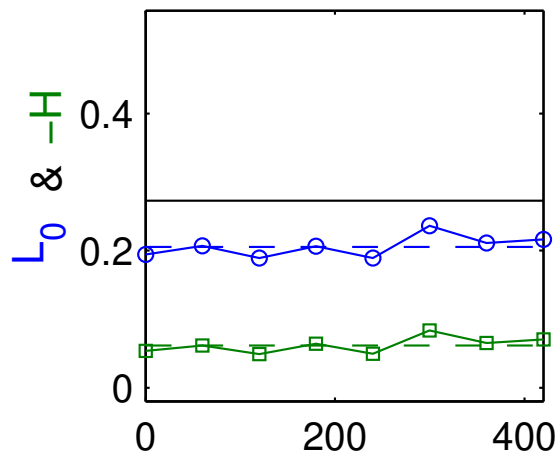
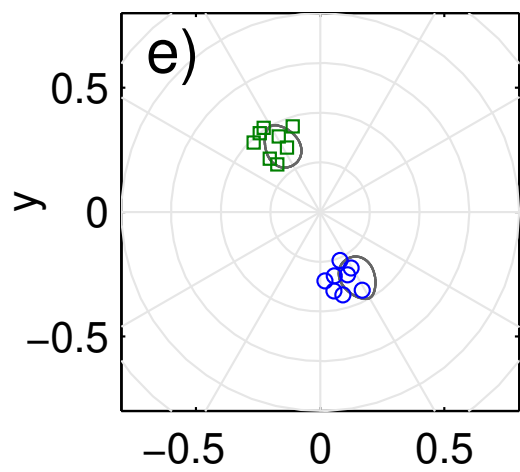
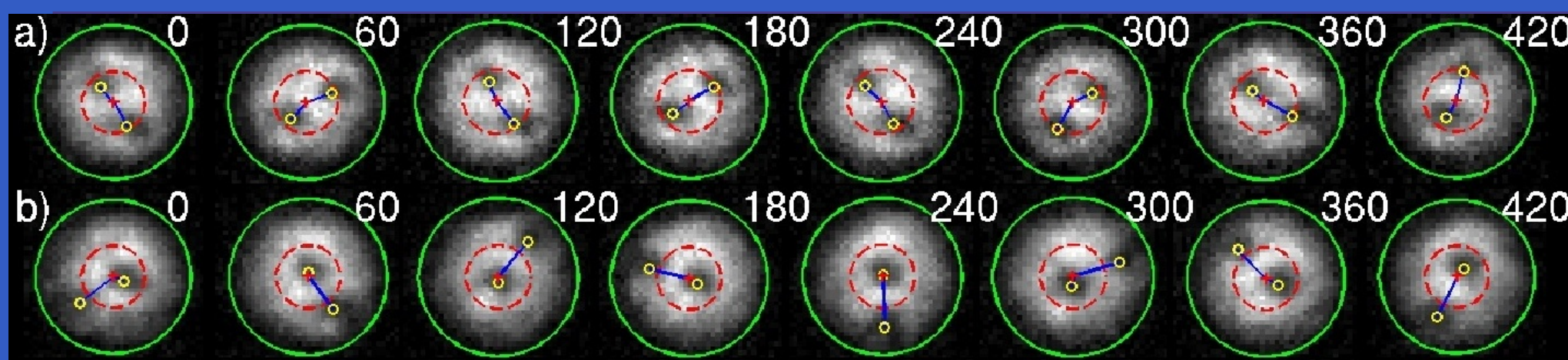
# Bif. of equilibria vs. angular momentum: 2 vortices

- Angular momentum  $L_0 = r_1^2 + r_2^2$  and  $\tan \phi = r_2/r_1$  (polar coord.)



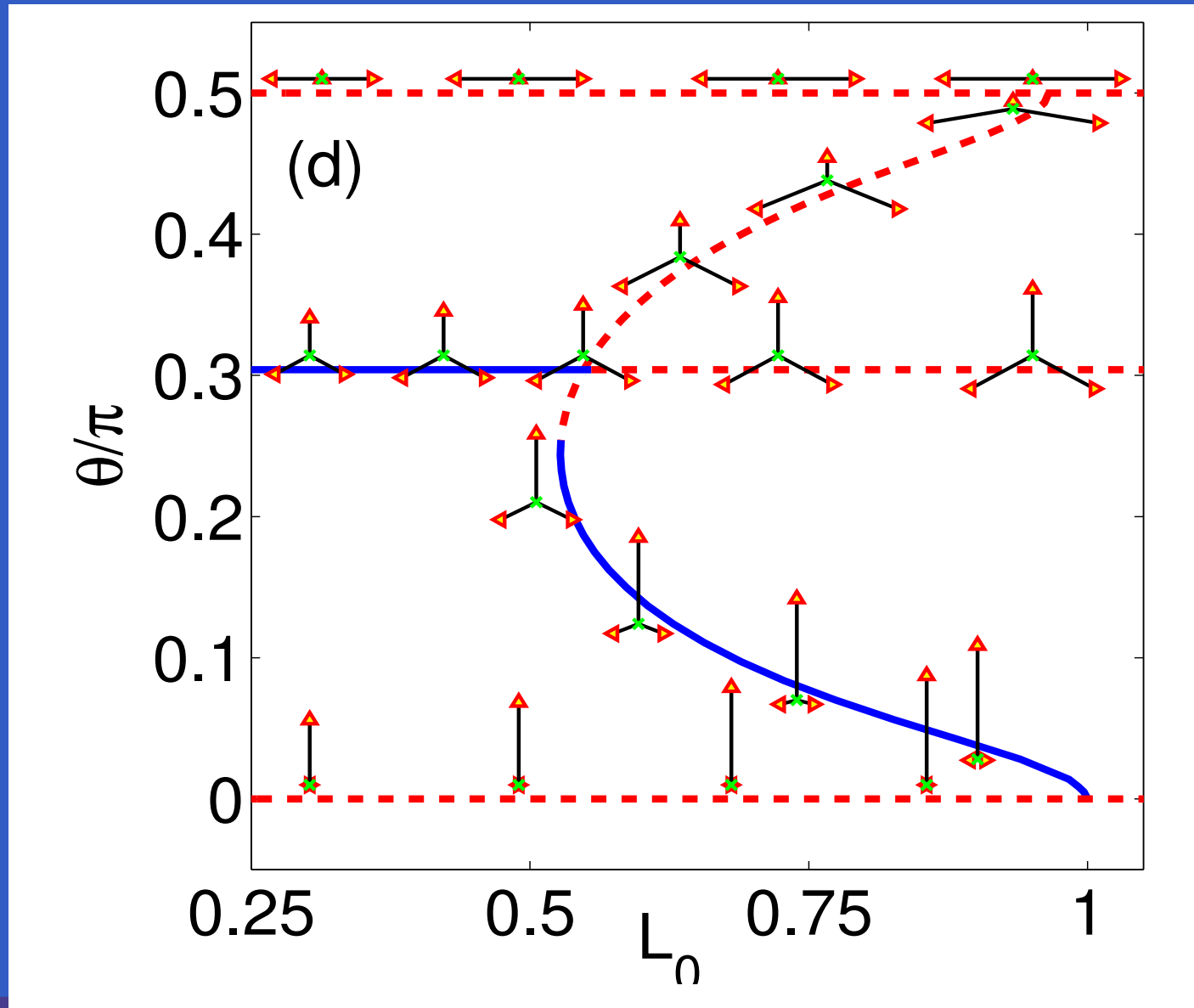


EXPERIMENTAL results:  
hunting for the pitchfork  
bifurcation (for  $N = 2$ )



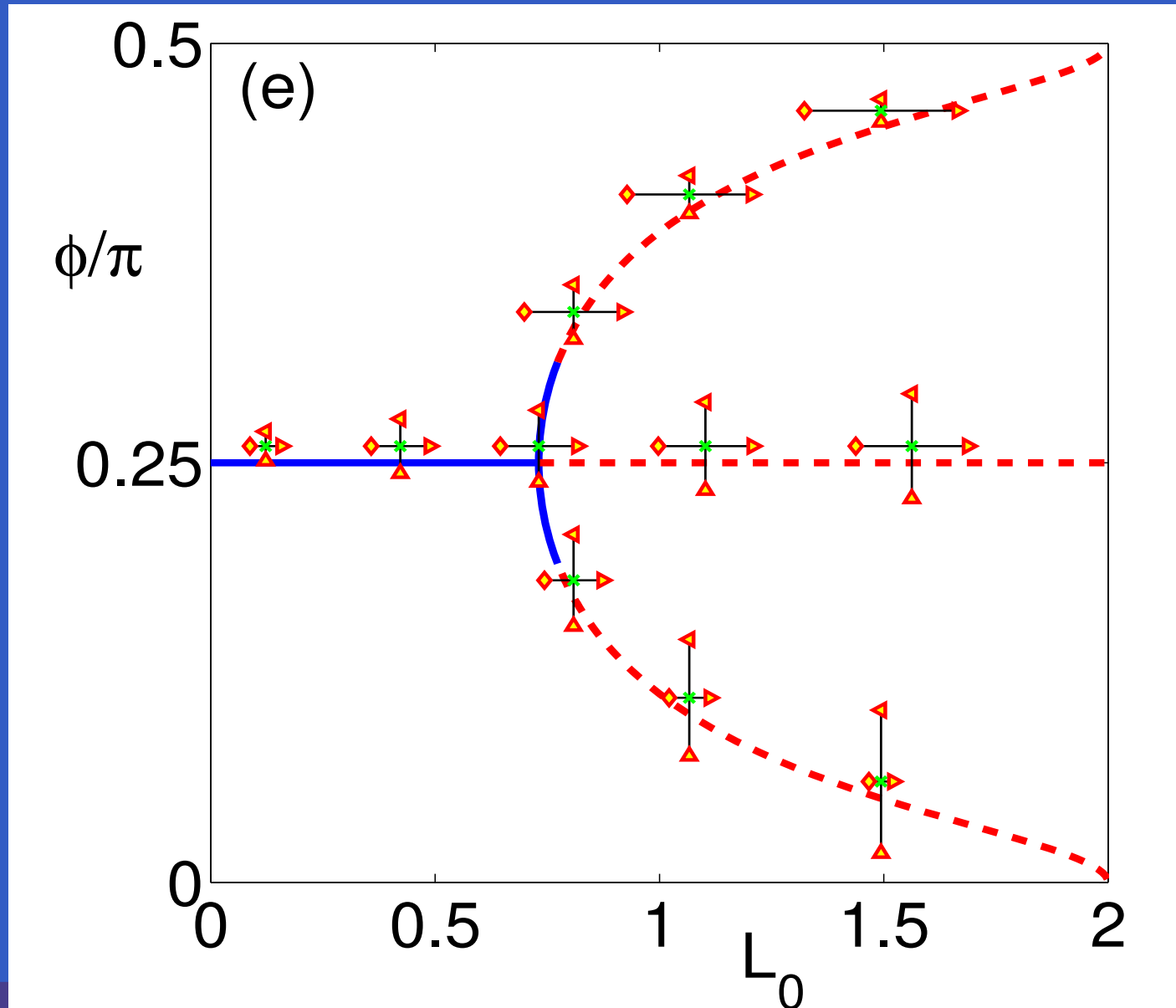
# Bif. of equilibria vs. angular momentum: 3 vortices

- Angular momentum  $L_0 = r_1^2 + r_2^2 + r_3^2$  and  $\tan \phi = r_2/r_1$  (polar coord.)



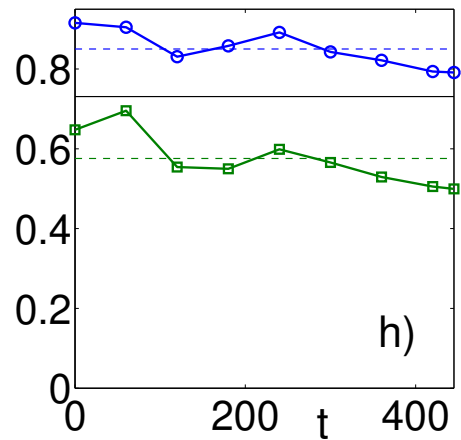
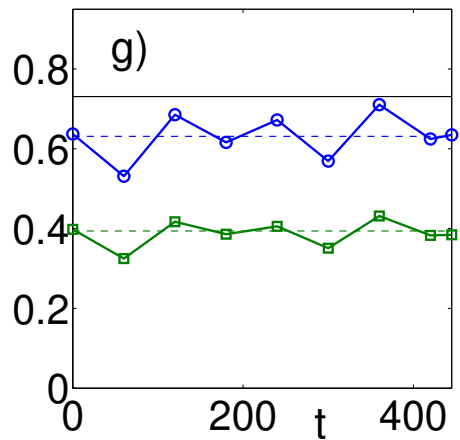
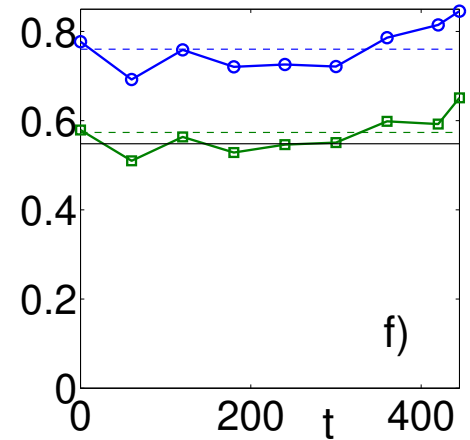
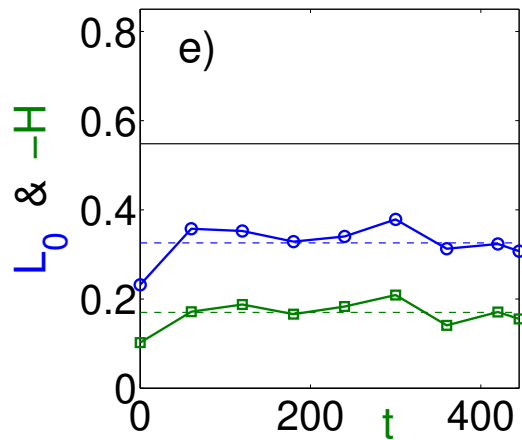
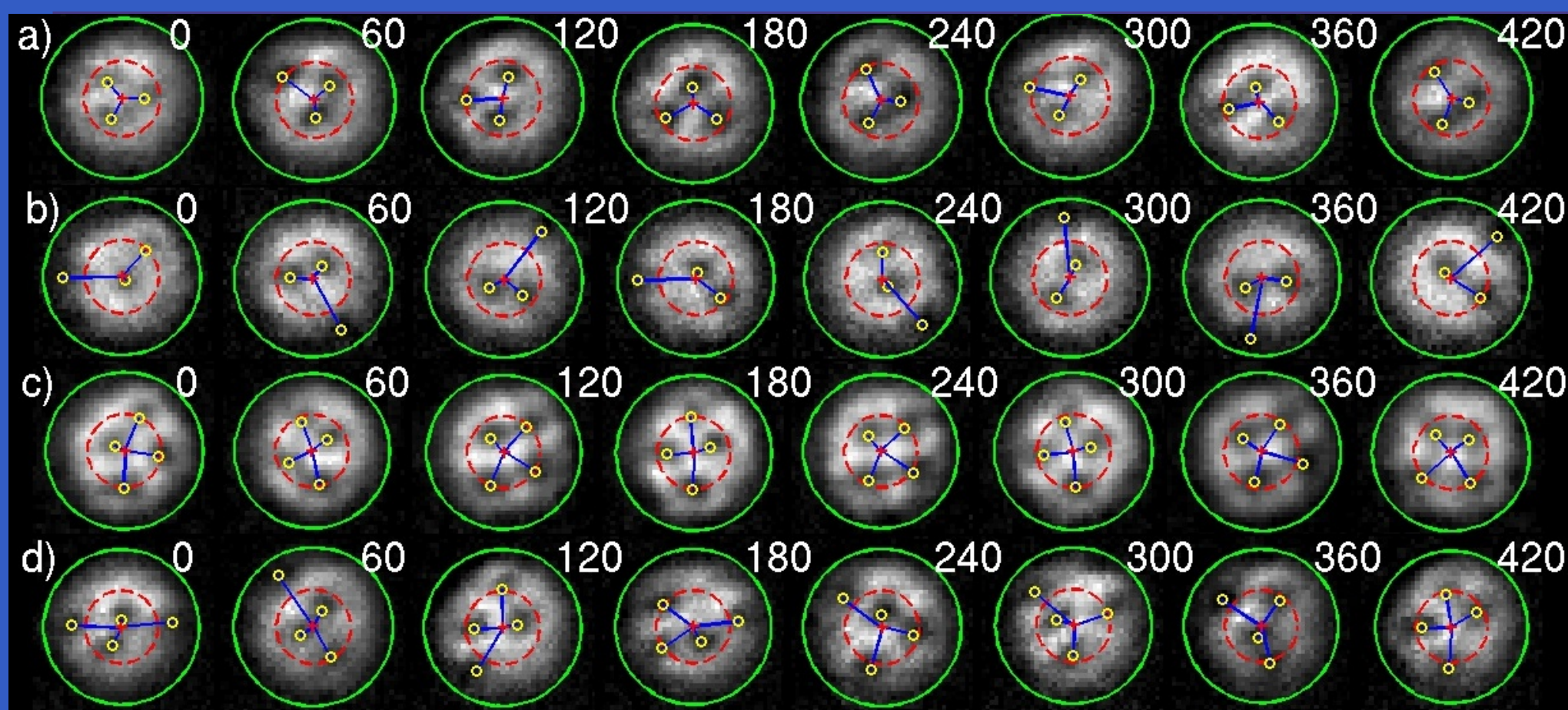
# Bif. of equilibria vs. angular momentum: 4 vortices

- Angular momentum  $L_0 = r_1^2 + r_2^2 + r_3^2 + r_4^2$



# Back to EXPERIMENTS: hunting for the pitchfork bifurcation (for $N > 2$ )





# EXTENSIONS:

loss & gain

anisotropy

non-Euclidean geometries

# Dissipation ( $T \neq 0$ ) $\Rightarrow$ Vortex outward spiralling.

- Phenomenological dissipation ( $\gamma > 0$ ):

$$(i - \gamma)u_t = -\frac{1}{2}\nabla^2 u + V(r)u + |u|^2 u.$$

- Energy loss:  $\frac{dE}{dt} = -2\gamma \iint |u_t|^2 dx dy$

$$\dots \Rightarrow \begin{cases} \dot{x} = -S\omega_{\text{pr}} y + \tilde{\gamma} \dot{y}, \\ \dot{y} = S\omega_{\text{pr}} x - \tilde{\gamma} \dot{x}, \end{cases}$$



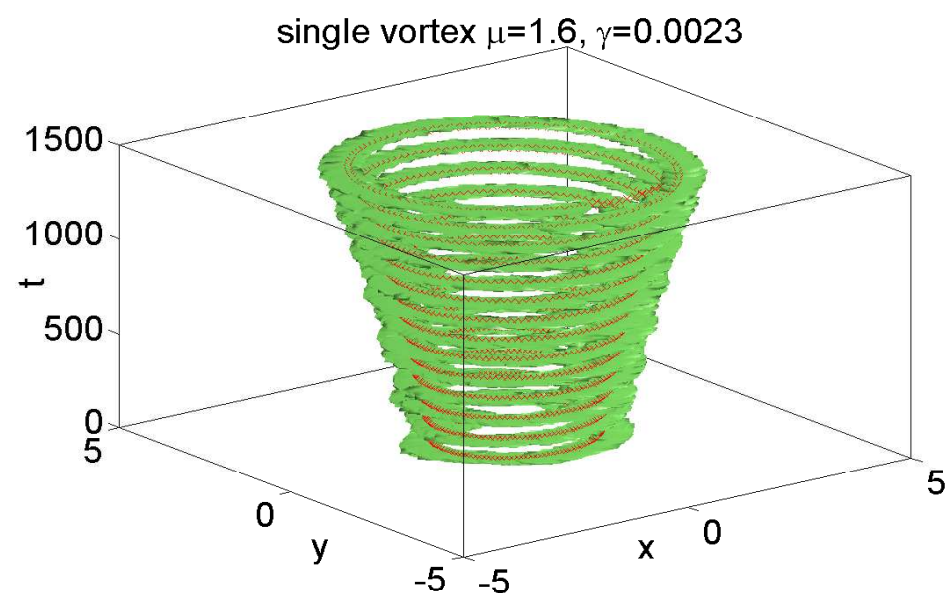
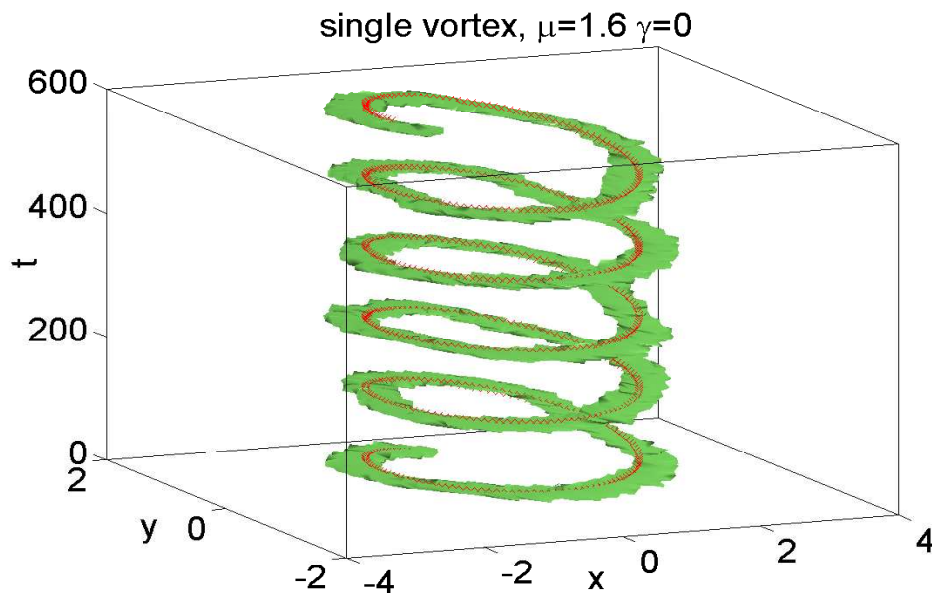
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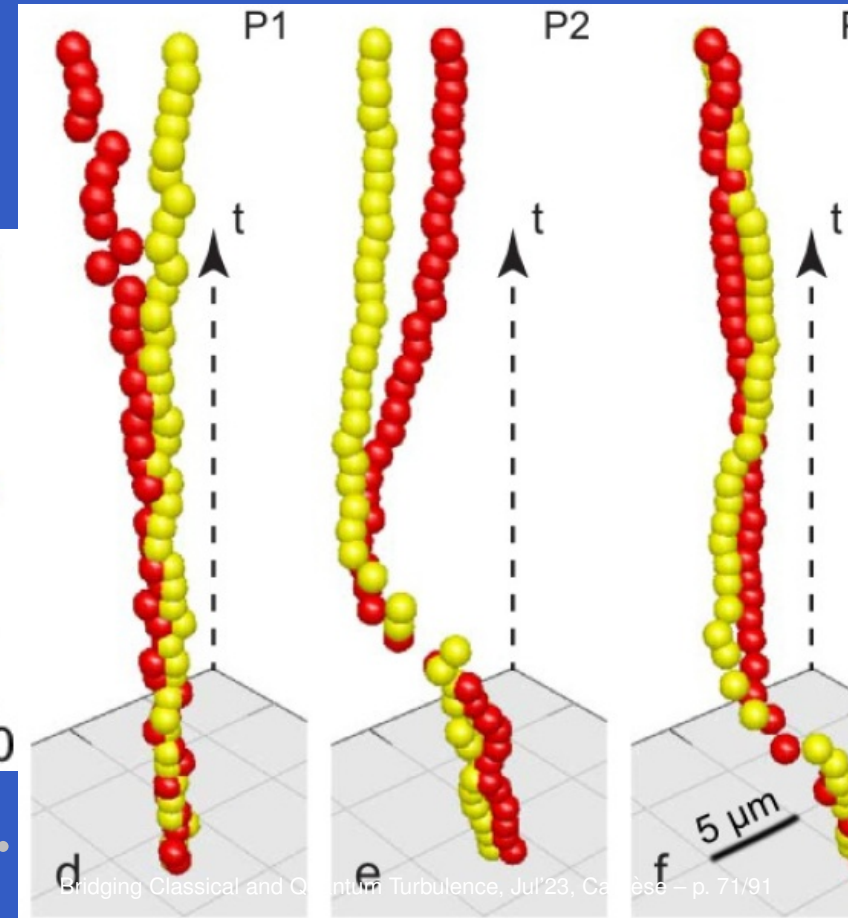
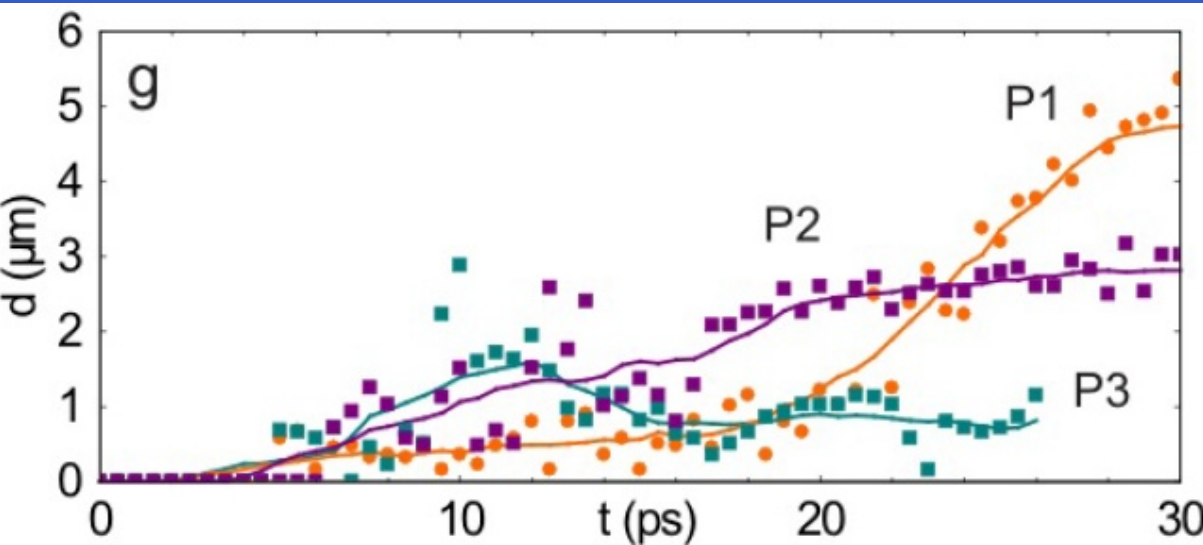


# Polariton BECs $\Rightarrow$ Loss + Gain $\rightarrow$ spiralling [Sanvitto].

- Experimental results in (spinor) exciton( $\phi$ )-polariton( $\psi$ ) condensates

$$i\frac{\partial\phi_{\pm}}{\partial\tau} = \left( -\epsilon\frac{1}{2}\nabla^2 - i\gamma_{\psi} + |\phi_{\pm}|^2 + \bar{\alpha}|\phi_{\mp}|^2 \right) \phi_{\pm} + \omega_R \psi_{\pm}$$

$$i\frac{\partial\psi_{\pm}}{\partial\tau} = \left( -\frac{1}{2}\nabla^2 - i\gamma_{\phi} \right) \psi_{\pm} + \omega_R \phi_{\pm} + \beta \left( \frac{\partial}{\partial X} \pm i\frac{\partial}{\partial Y} \right)^2 \psi_{\mp} + F_{\pm},$$



# Anisotropy $\rightarrow$ complex periodic orbits

$N$  vortices with positions  $(x_i, y_i)$  in an anisotropic trap

$$\begin{aligned}\dot{x}_k &= -s_k Q \omega_y^2 y_k + \frac{B}{2} \sum_{j \neq k} s_j \frac{y_j - y_k}{\rho_{jk}^2}, \\ \dot{y}_k &= s_k Q \omega_x^2 x_k - \frac{B}{2} \sum_{j \neq k} s_j \frac{x_j - x_k}{\rho_{jk}^2},\end{aligned}$$

where  $\rho_{jk}^2 = (x_j - x_k)^2 + (y_j - y_k)^2$  and (after rescaling)  $Q \omega_x^2 = 1$  and  $Q \omega_y^2 = 1 + \epsilon$ .

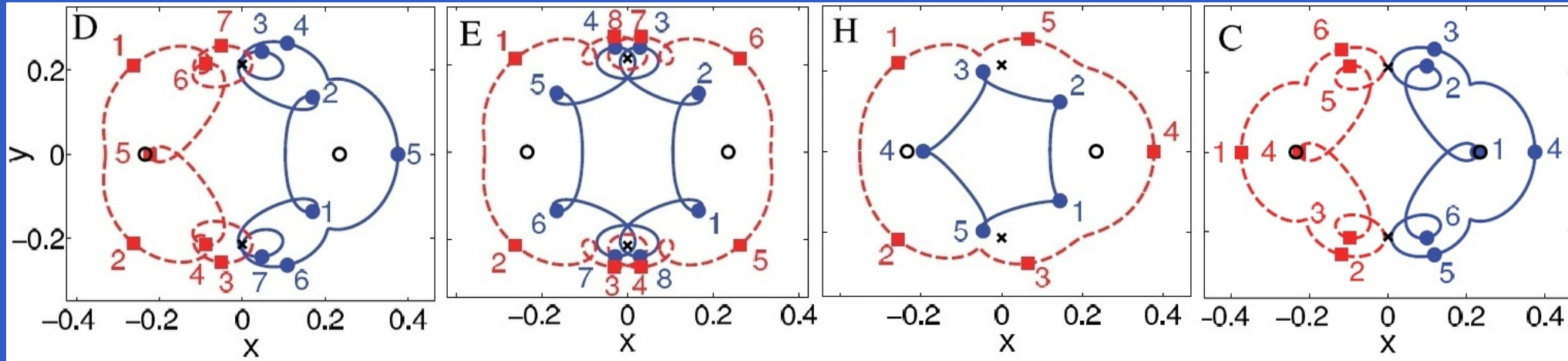
TWO opposite charge vortices (dipole)  $\rightarrow$  fixed points are:

$$\vec{P}_x = \pm \sqrt{\frac{B}{4(1+\epsilon)}} (0, 0, 1, -1) \quad \text{and} \quad \vec{P}_y = \pm \frac{\sqrt{B}}{2} (1, -1, 0, 0).$$

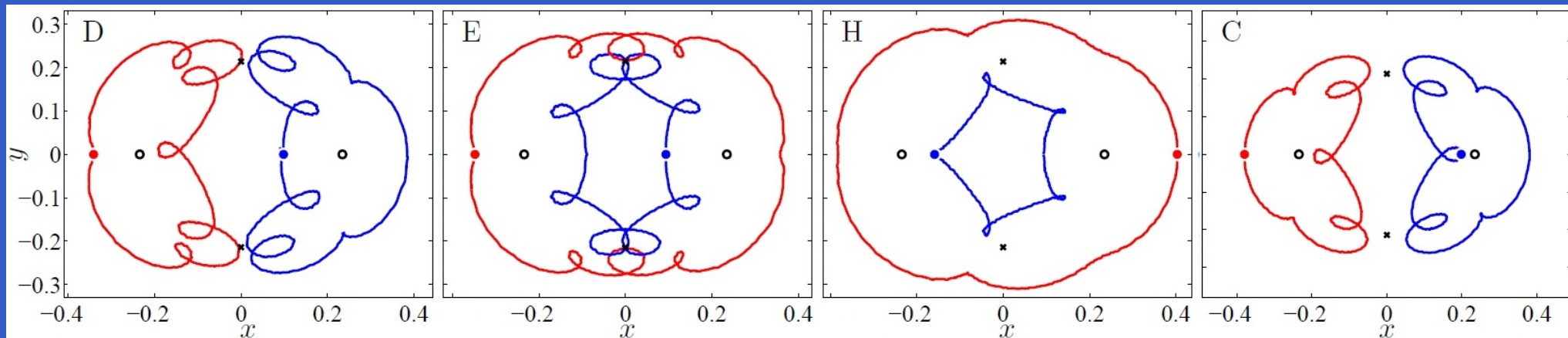
Reducing the system to a pendulum coupled to harmonic oscillator  
 $\Rightarrow$  conserved quantities  $\Rightarrow$  approx Poincaré map for periodic orbits  
 $\Rightarrow$  determination of ICs leading to complicated periodic orbits.

# Anisotropy $\rightarrow$ complex periodic orbits $\rightarrow$ ODE vs PDE

ODE:



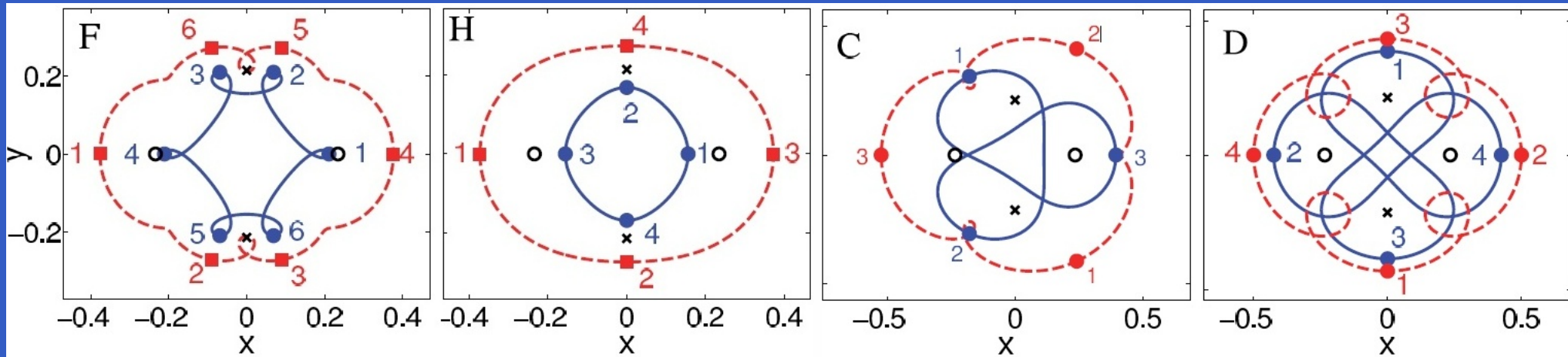
PDE:



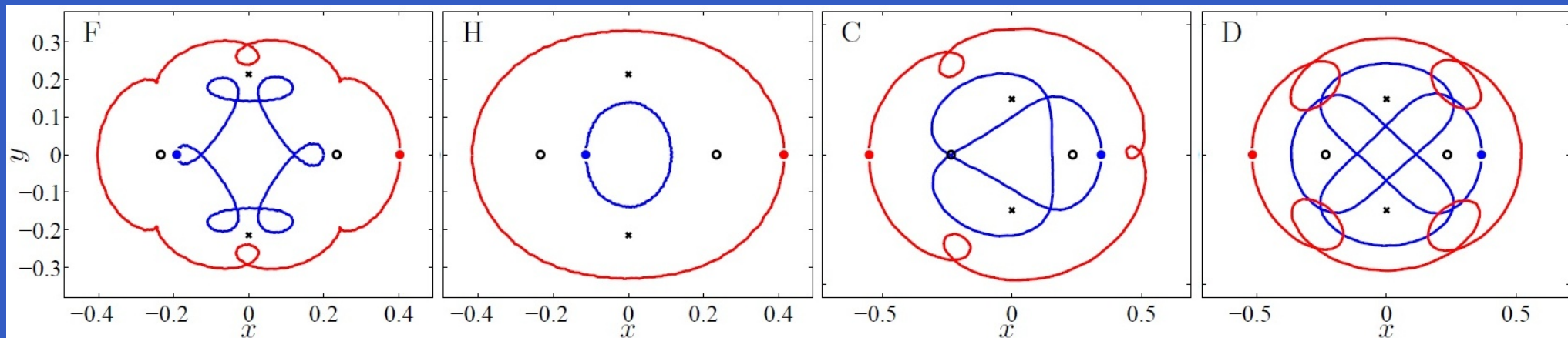


# Anisotropy $\rightarrow$ complex periodic orbits $\rightarrow$ ODE vs PDE

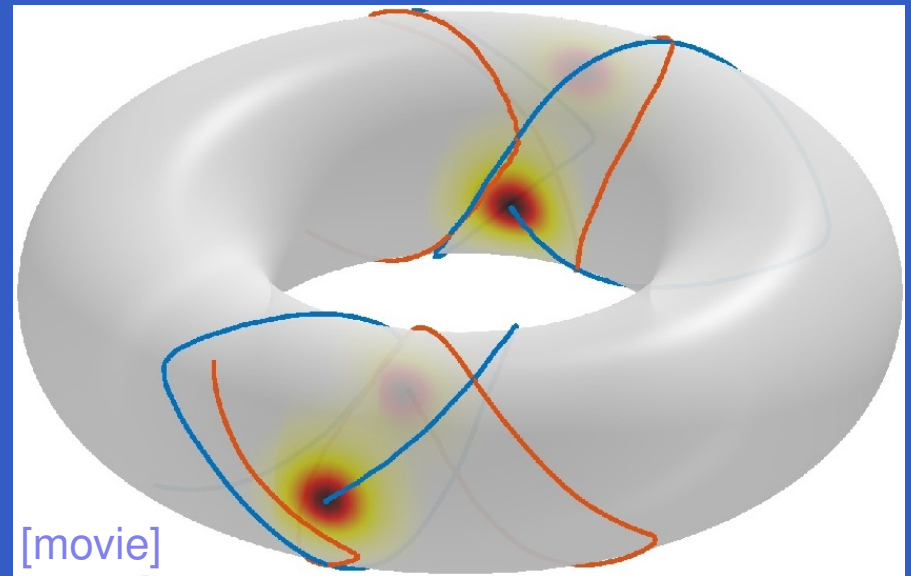
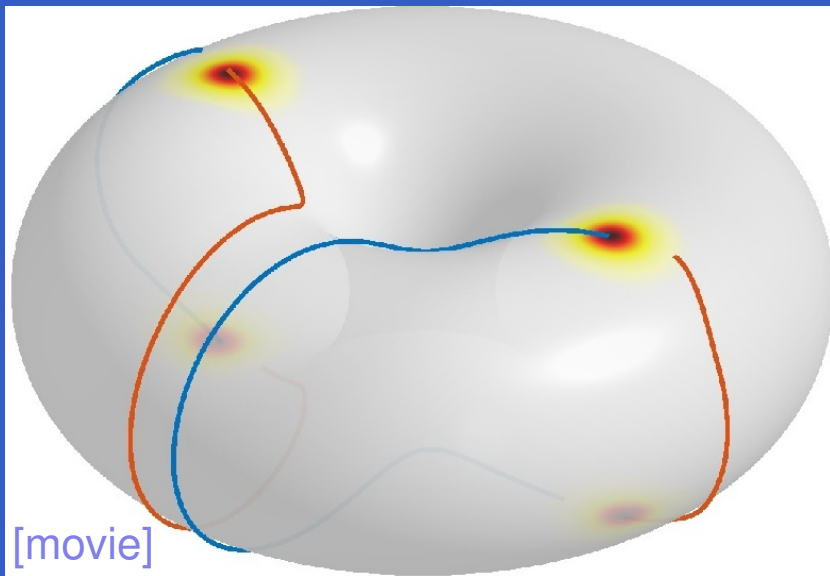
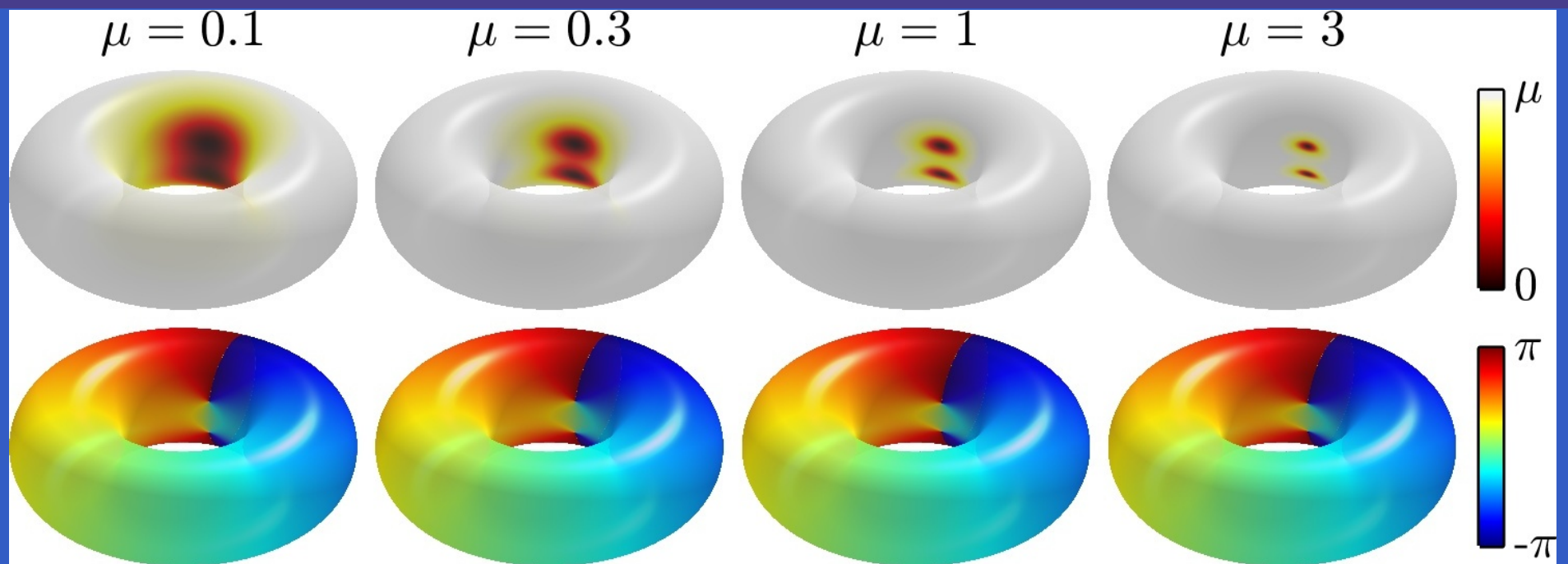
ODE:



PDE:



# Vortex dynamics in non-Euclidean geometries



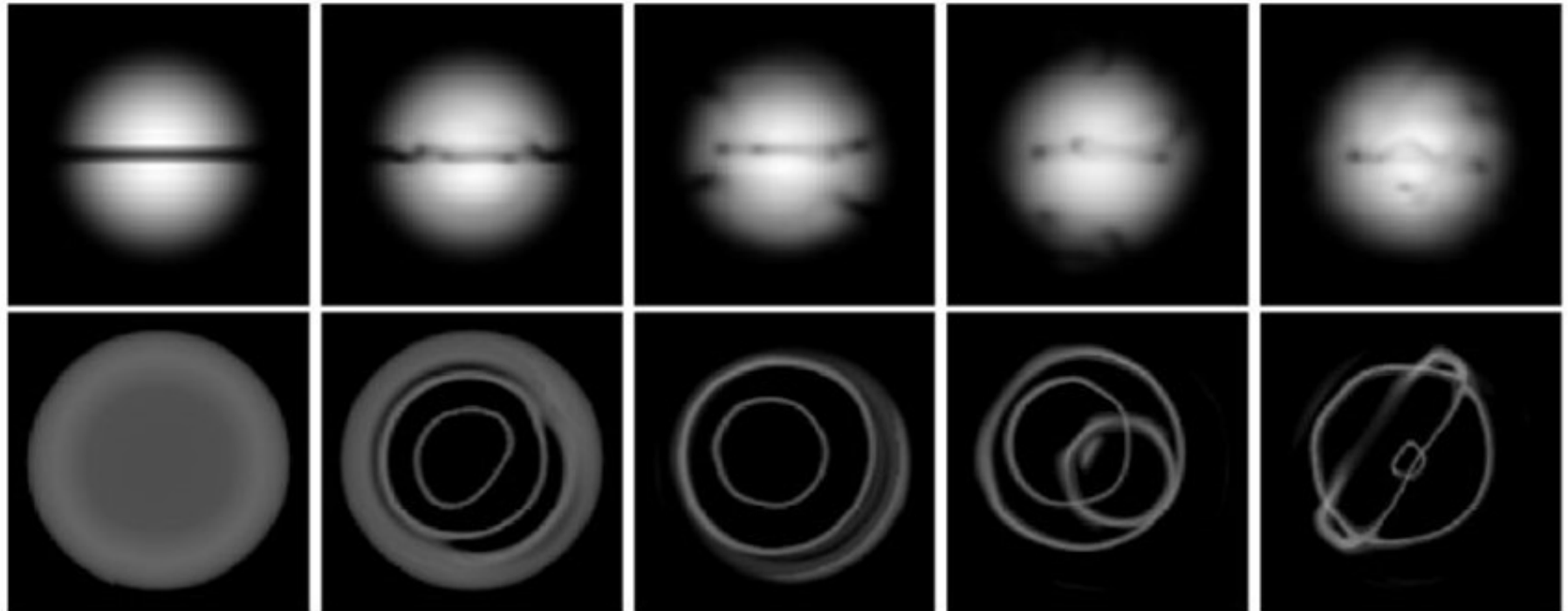
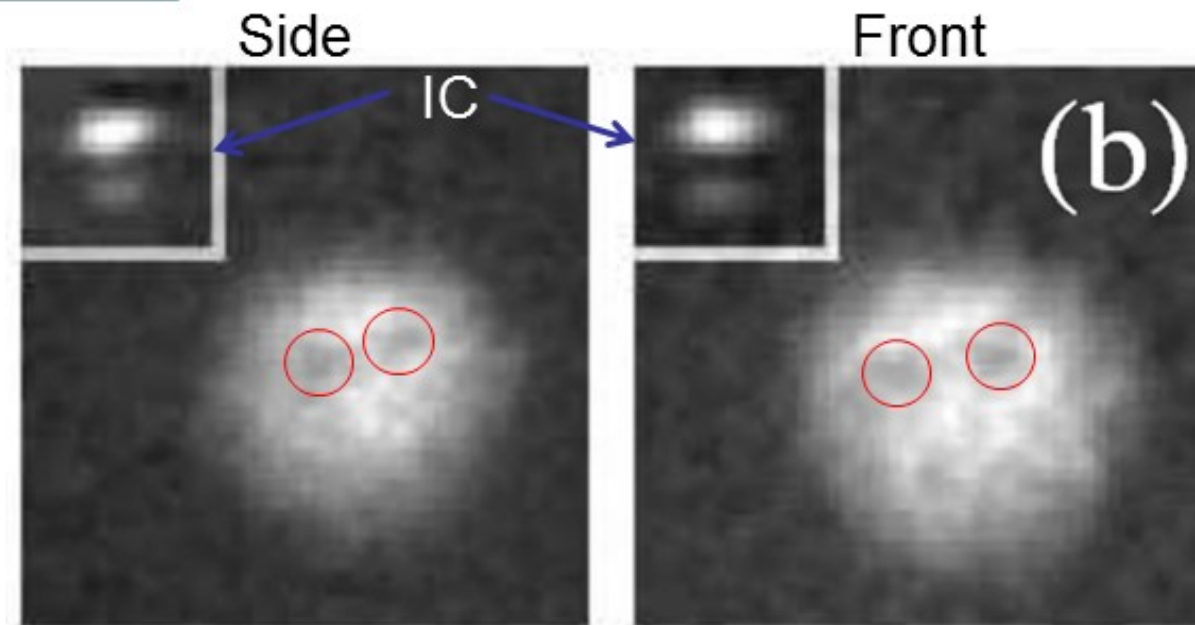
# Vortex Rings in 3D BECs

## Physical Motivation

- VRs observed in (many) experiments:

PRL **86** (2001) 2926. "Watching Dark Solitons Decay into VRs in a BEC", B.P. Anderson et al.

## Vortex Rings BECs





# Particle picture: NLS + Madelung = superfluid

- NLS + Madelung transformation ( $u = \sqrt{\rho} e^{i\phi}$ )
- Def.  $v = \nabla\phi \rightarrow$  fluid velocity
- NLS  $\rightarrow$  Non-viscous Eulerian fluid + “quantum pressure”
- Quant. press.  $\propto \nabla^2 \sqrt{\rho} / \sqrt{\rho} \rightarrow$  negligible away from vortex cores

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- Quant. press.  $\propto \nabla^2 \sqrt{\rho} / \sqrt{\rho} \rightarrow$  negligible away from vortex cores
- Let us borrow technology from classical fluids: Biot-Savart law:

$$\mathbf{u}_v(\mathbf{x}) = \frac{1}{4\pi} \int \frac{\boldsymbol{\omega}(\mathbf{x}') \times (\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^3} d\mathbf{x}', \quad (3)$$

where  $\omega$ =vorticity and integral is computed over all of space.

- Consider a vorticity tube with circulation  $\kappa$  along a closed curve  $\mathbf{R}$  parametrized by  $\ell$ :

$$\mathbf{u}_v(\mathbf{x}) = \frac{\kappa}{4\pi} \oint \frac{\mathbf{s} \times (\mathbf{x} - \mathbf{R}(\ell))}{|\mathbf{x} - \mathbf{R}(\ell)|^3} d\ell \quad (4)$$

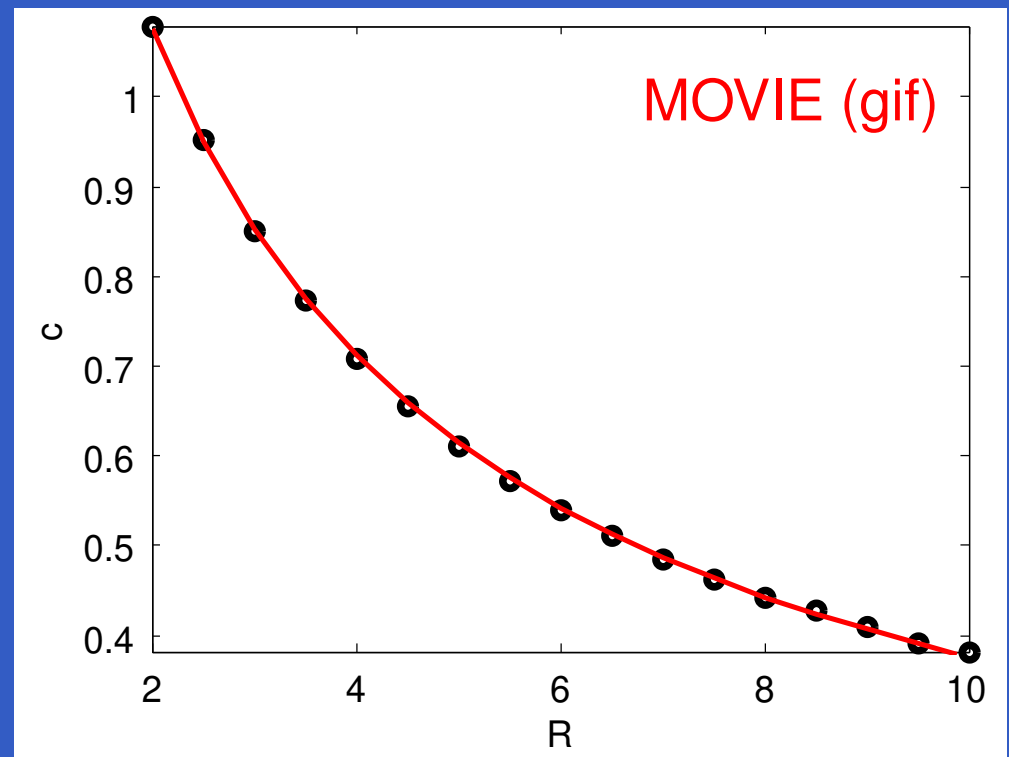
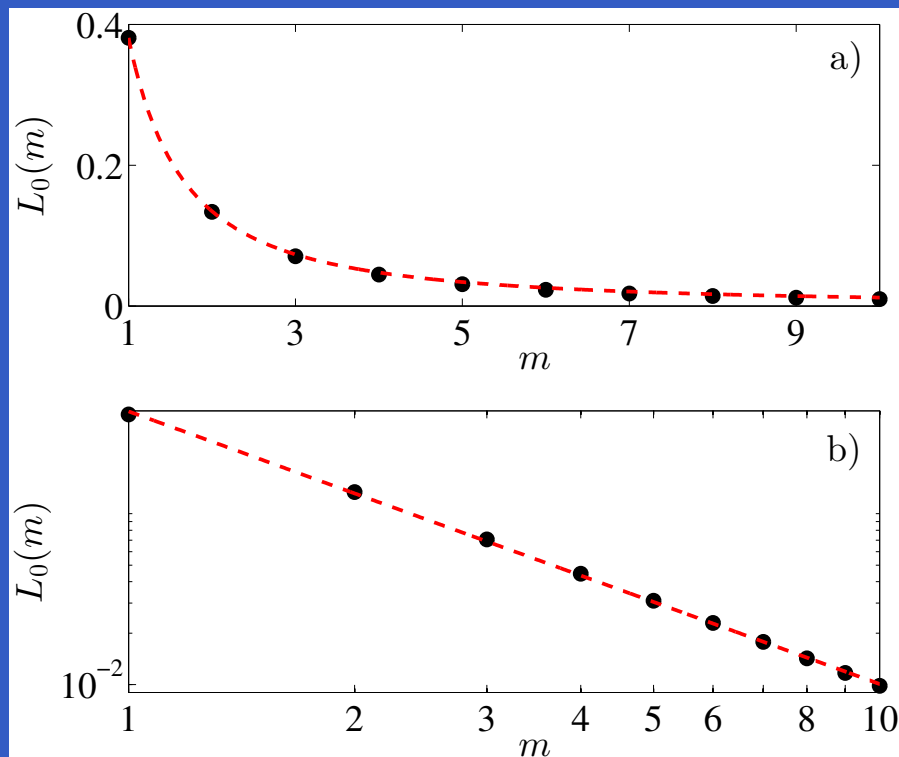
# Single VR $\rightarrow$ self-induced velocity

- The Biot-Savart law diverges for self-induced velocity
- $\rightarrow$  use local induction approximation (matched asymptotics)

- $\rightarrow$  Self-induced velocity:  $c \approx -\frac{m}{2R} \left( \ln \frac{8R}{r_c} + L_0(m) - 1 \right)$ ,

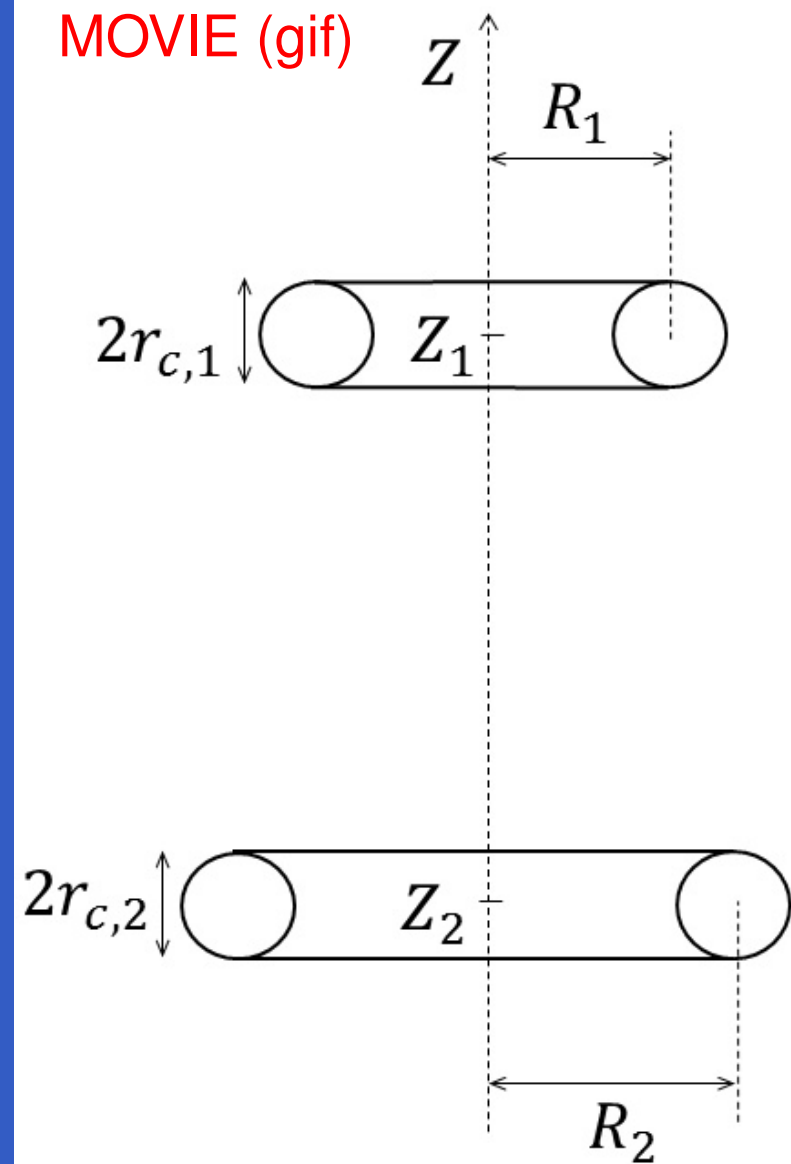
radius:  $R$ , charge:  $m$ , core size (healing length):  $r_c$

$L_0(m) \approx L_0(1) m^{-3/2}$ : vortex core parameter (numerically)



# Particle picture: two VRs ( $V_{\text{trap}} = 0$ ) $\rightarrow$ leapfrogging

MOVIE (gif)



# Particle picture: two VRs ( $V_{\text{trap}} = 0$ ) $\rightarrow$ leapfrogging

Biot-Savart law & local induction approx:

$$\dot{Z}_i = \frac{\kappa_i}{4\pi R_i} \left( \ln \frac{8R_i}{r_{c,i}} - C \right) + \frac{1}{\kappa_i R_i} \frac{\partial U}{\partial R_i},$$

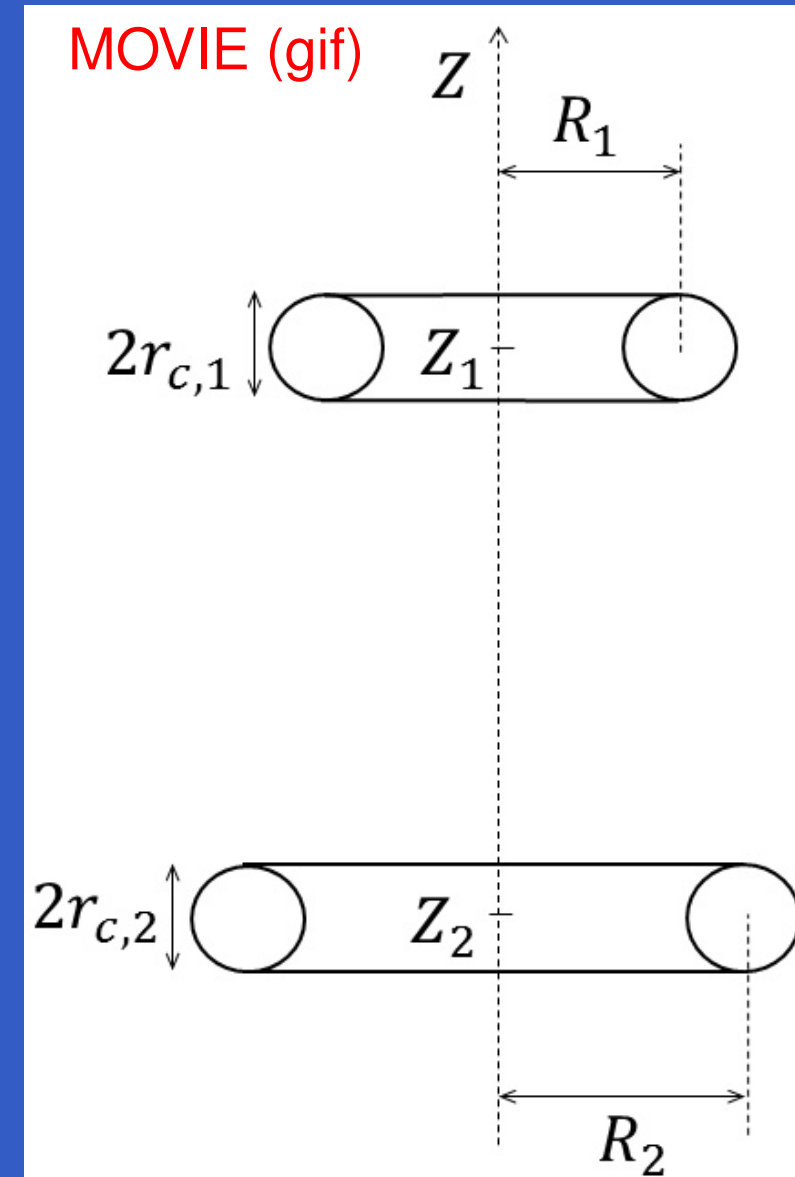
$$\dot{R}_i = -\frac{1}{\kappa_i R_i} \frac{\partial U}{\partial Z_i},$$

where

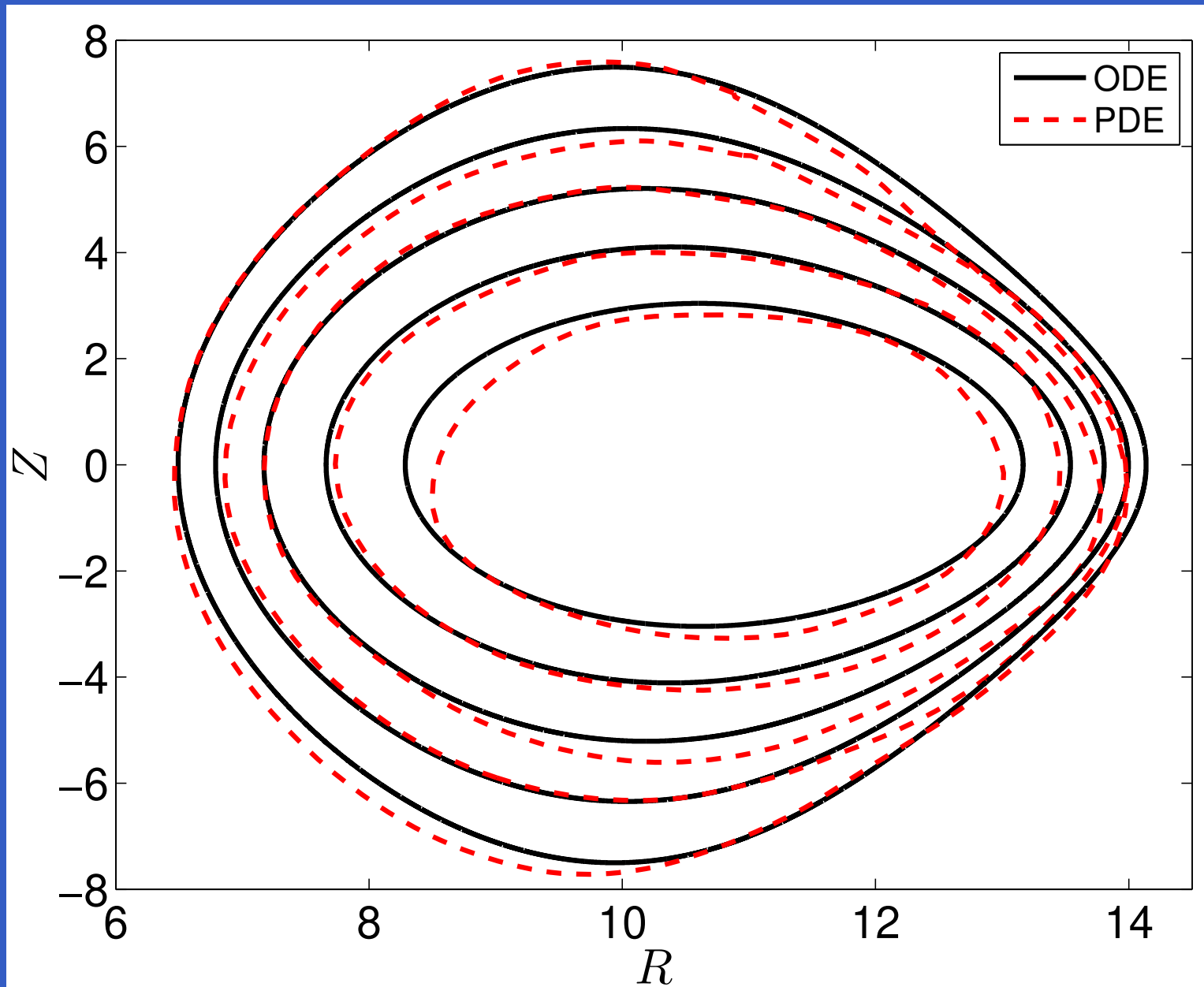
$$U = \frac{1}{2\pi} \sum_{i=1}^N \sum_{j>i}^N \kappa_i \kappa_j I_{ij},$$

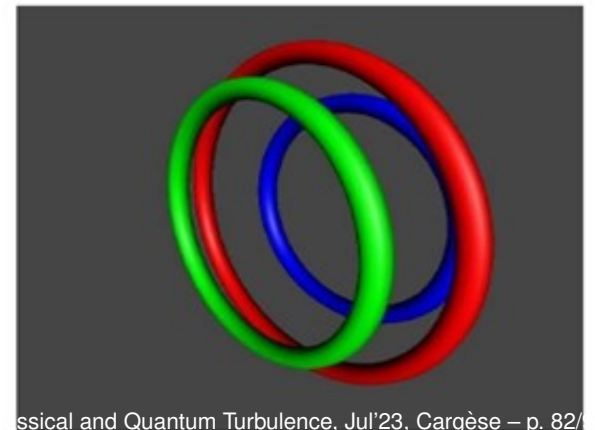
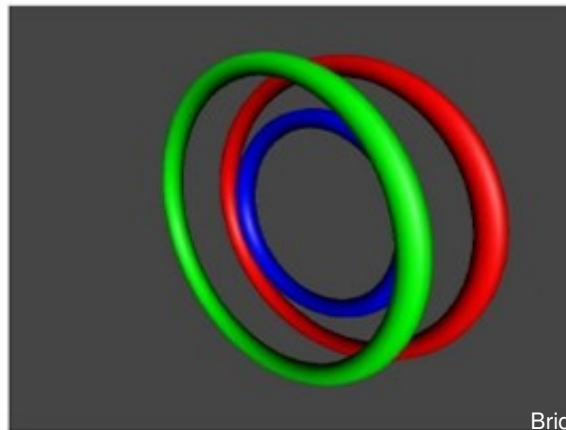
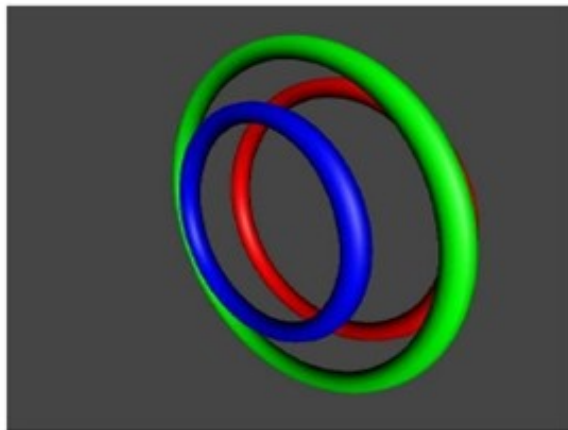
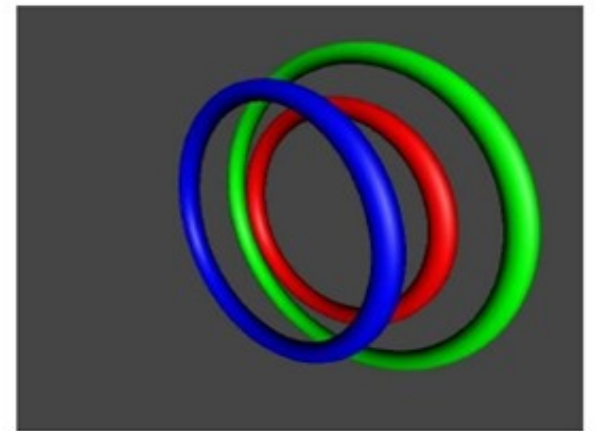
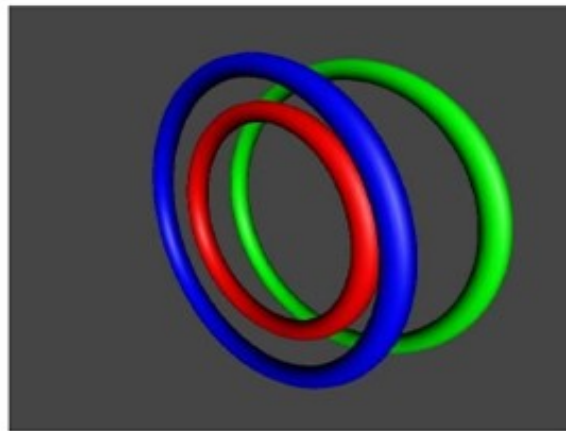
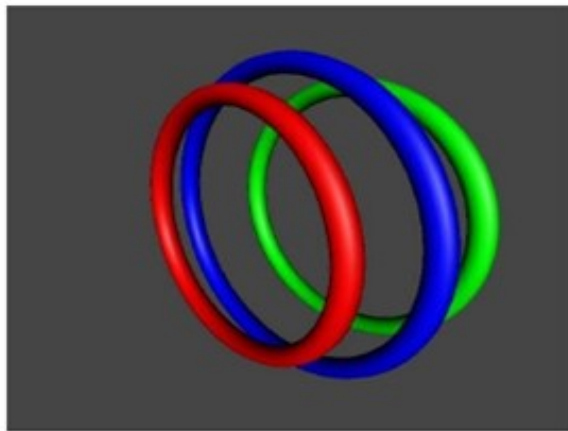
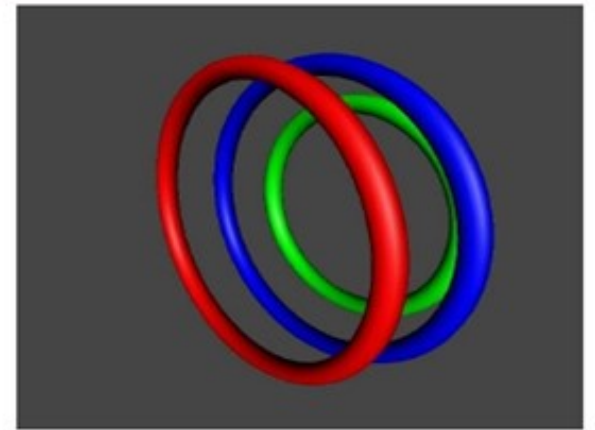
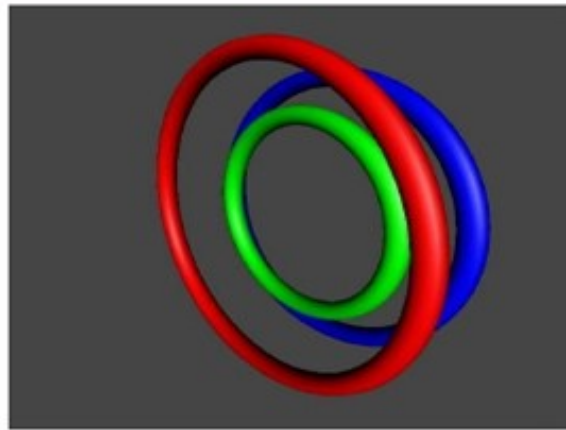
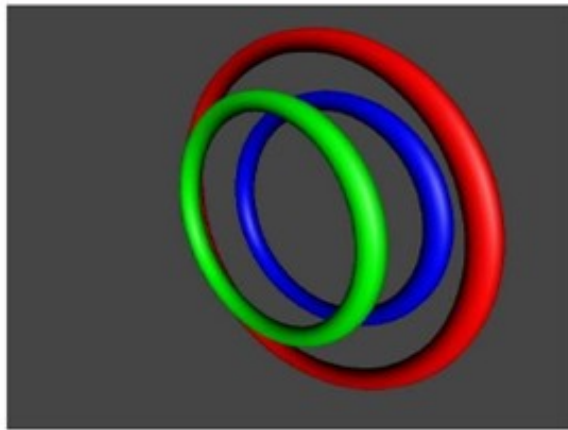
and

$$I_{ij} = \int_0^\pi \frac{R_i R_j \cos \theta d\theta}{\sqrt{(Z_i - Z_j)^2 + R_i^2 + R_j^2 - 2R_i R_j \cos \theta}}$$



# Leapfrogging of VRs: PDE vs ODE reduction





# Adding $V_{\text{trap}} = \text{harmonic} \rightarrow \text{Particle picture}$

- naïve approach;
  - $\rightarrow$  just add VR-VR and VR- $V_{\text{trap}}$  interactions
  - $\rightarrow$  it does NOT work :(
  - VR-VR interaction (see previous)  $\rightarrow H$  canonical variables  $(r_i^2, z_i)$ .
  - VR- $V_{\text{trap}}$  interaction (pert./variational)  $H$  canonical var.  $(r_i, z_i)$ .



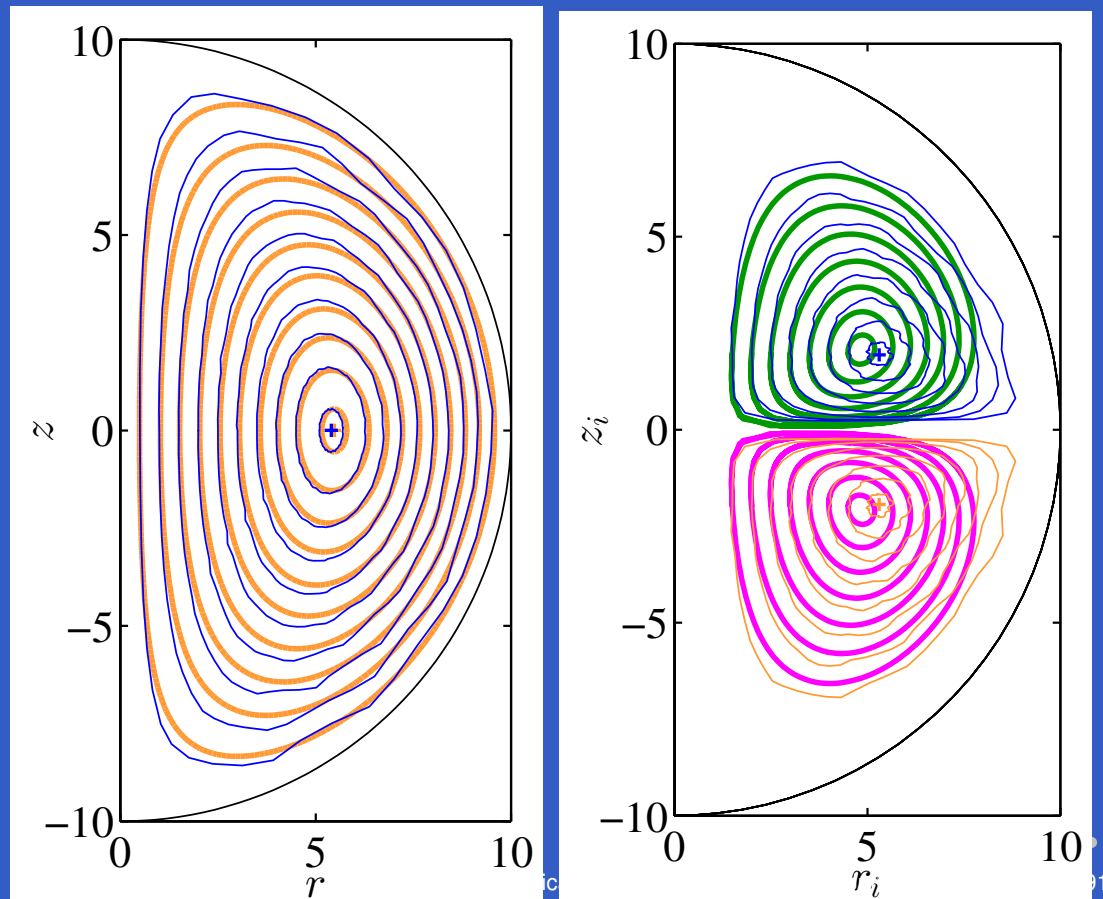
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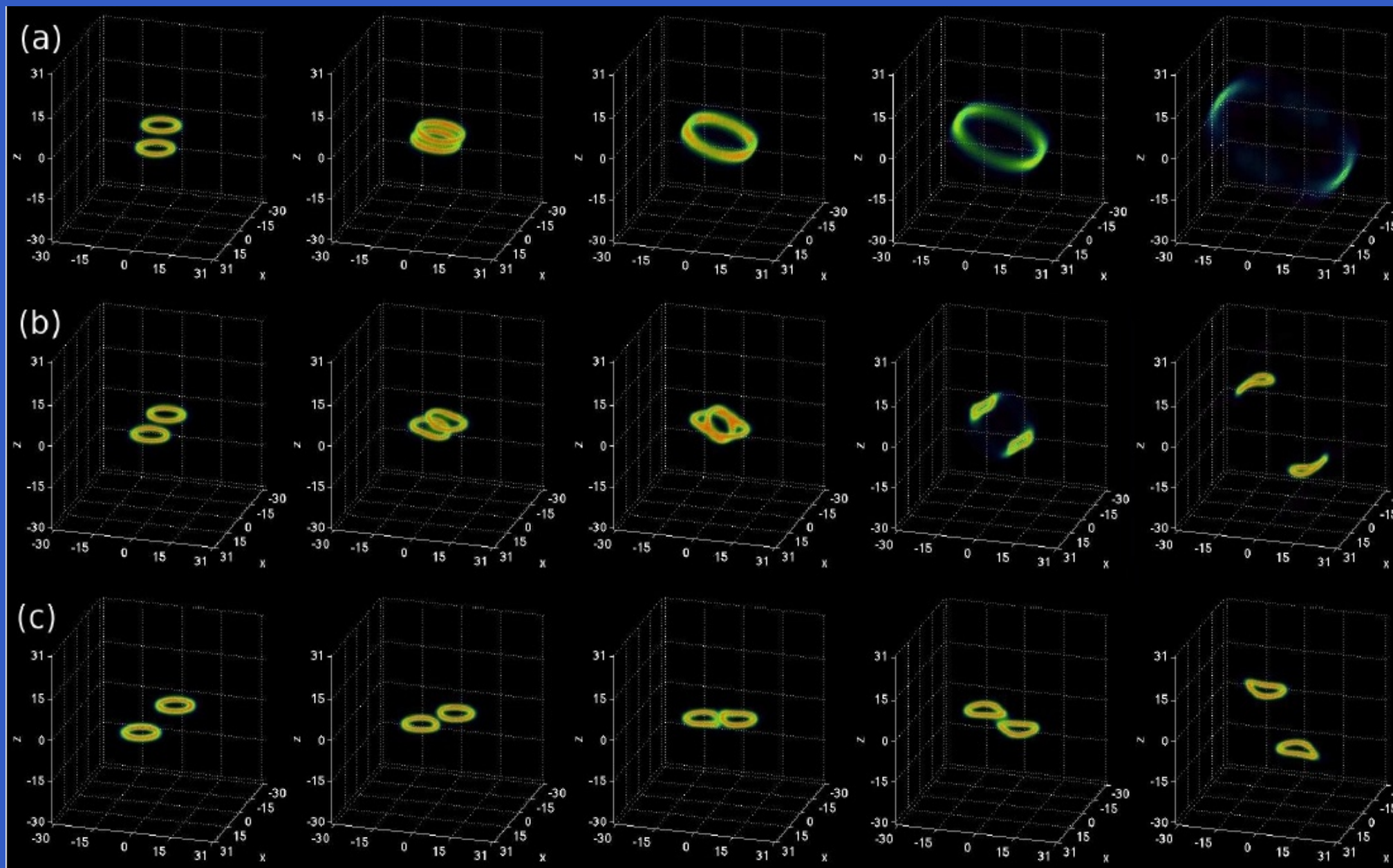
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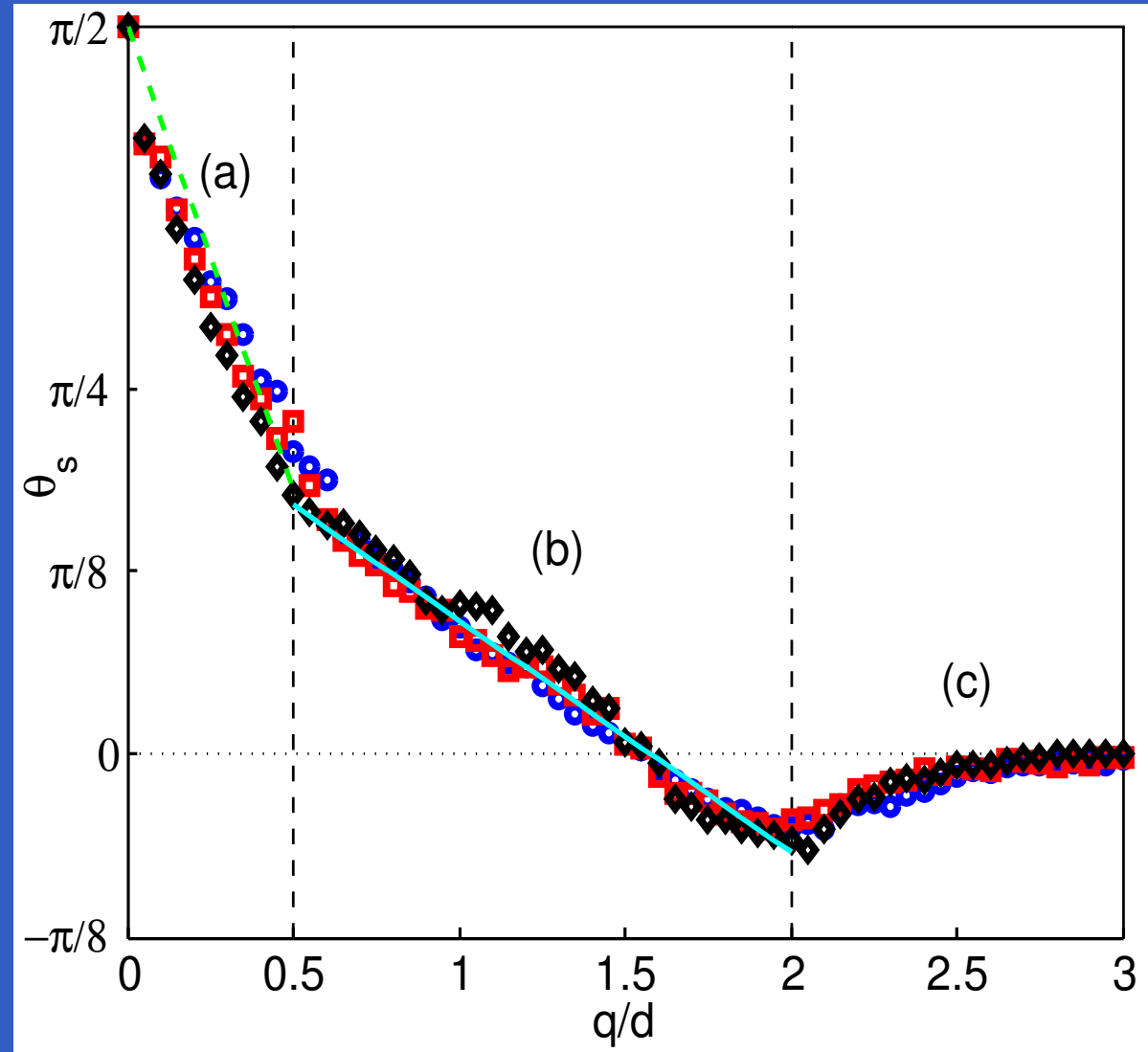
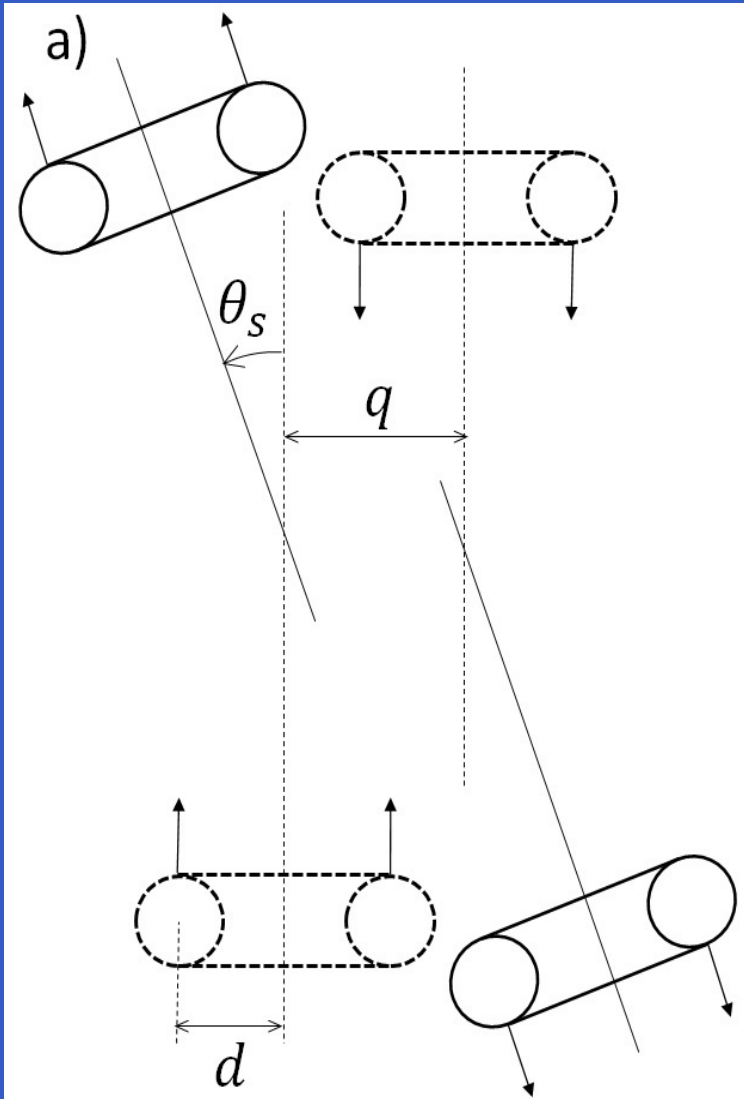
# VR scattering scenarios: [Head-on movie] [Scattering movie]

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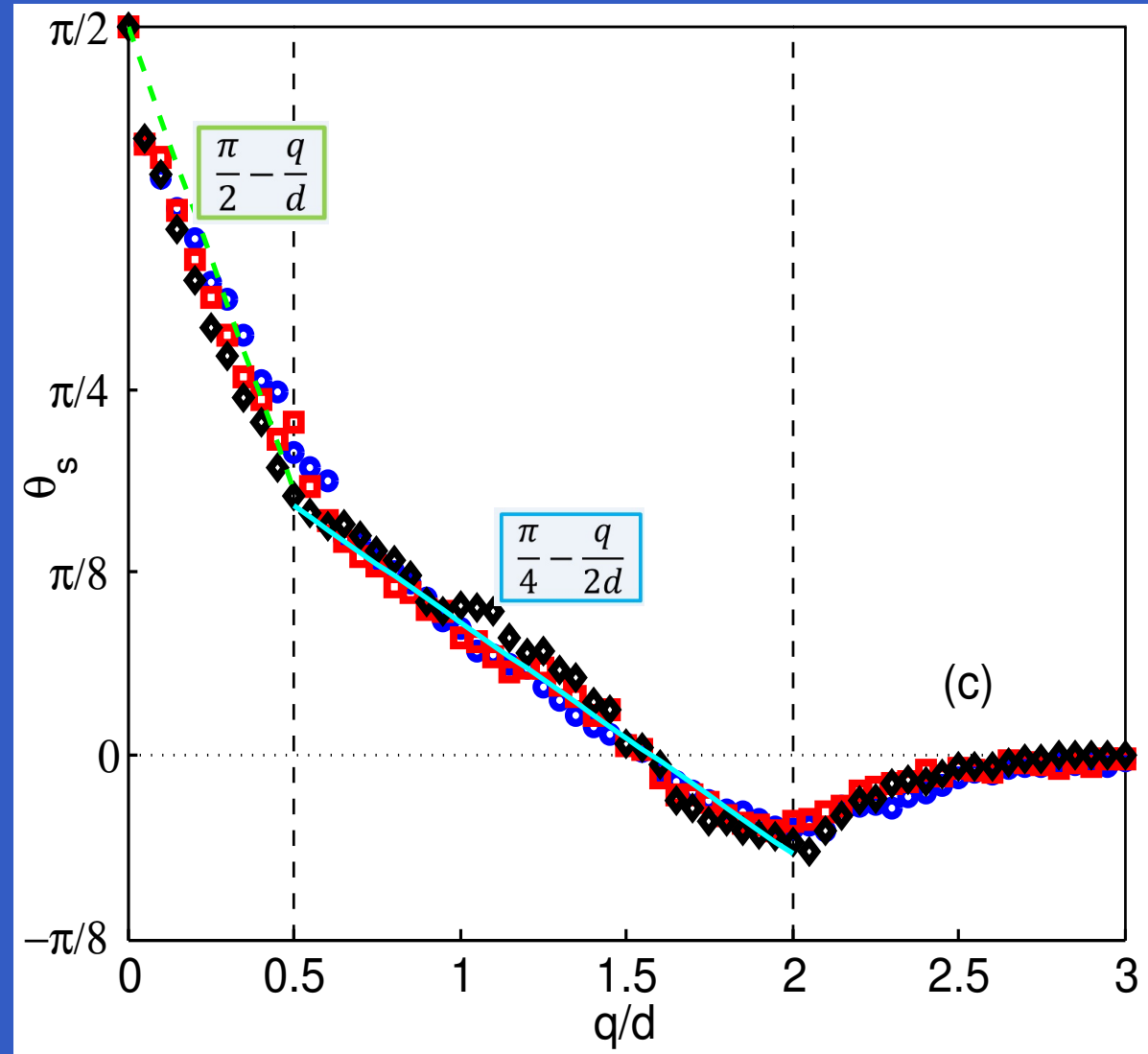
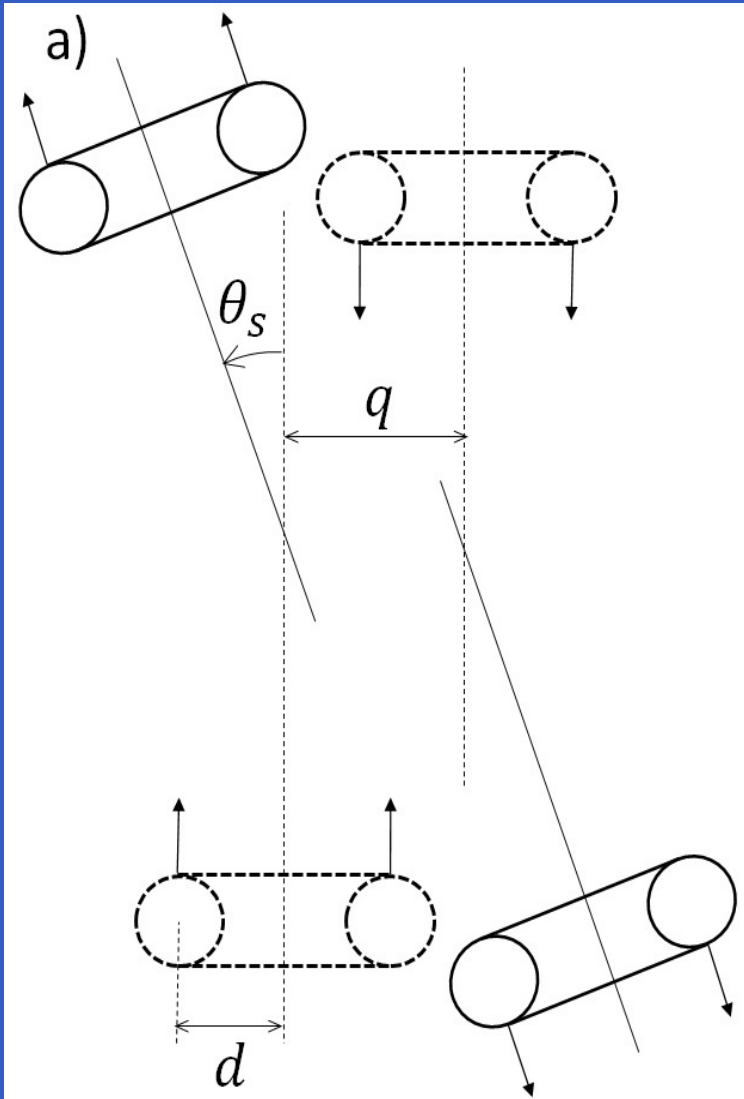
[Head-on movie] [Scattering movie]



# VR scattering angle (numerical results):



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- Outlook and work in progress:
  - Vortices in non-Euclidean geometries (elliptical torus  $\rightarrow$  chaos!)
  - VR interactions for general angles
  - Introduce internal Kelvin modes ( $\rightarrow$  changes in self-velocity)
  - Generalizations to two-component BECs
  - Connect with experiments

END ...

Merci !

# BECs: Theory and Experiment [Springer, 2008]

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Theory and Experiment

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**D. J. Frantzeskakis** is a Professor in the Department of Physics at the University of Athens, Greece. His research interests include nonlinear waves and solitons, with applications in various physical contexts. He has supervised seven Ph.D. theses, has co-organized several international symposia, and was a guest editor of two international journals. He has authored or co-authored more than 200 peer-reviewed publications, including four invited review papers, and he has co-edited four books.

**R. Carretero-González** is a Professor of Applied Mathematics at San Diego State University (SDSU). His research focuses on spatio-temporal dynamical systems, nonlinear waves, and their applications. He is the co-founder and co-director of the Nonlinear Dynamical Systems (NLDS) group at SDSU. He has received multiple NSF grants and has published more than 100 peer-reviewed manuscripts, including three co-authored/edited books. He is an active advocate of the dissemination of science, continuously delivers engaging presentations at local high schools and science festivals, and helps design museum exhibits.

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3600 Market Street, 6th Floor

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+1-215-382-9800 • Fax +1-215-386-7999

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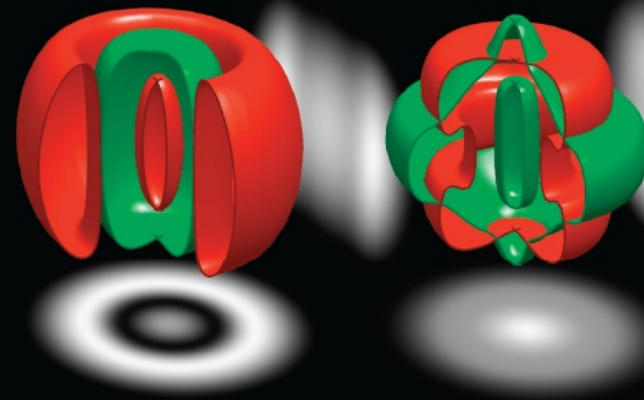
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  - MATH-693B : Numerical PDEs

- Fall Year 2:
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  - MATH-639 : Nonlinear Waves
  - MATH-797 : Research

- Spring Year 2:
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