Complex structures in quantum hydrodynamics

Andrew Baggaley – Newcastle University

George Grimes, Carlo Barenghi, Jason Laurie, Luca Galantucci
Classical Vorticity

\[ \omega = \nabla \times \mathbf{u} \]
\[ \Gamma = \phi_c \mathbf{v} \cdot d\mathbf{l} \in \mathbb{R} \]

\[ \Gamma = \phi_c \mathbf{v} \cdot d\mathbf{l} = \frac{2\pi \hbar}{m} n \]

Paoletti et al., 2008
Complex structures in quantum hydrodynamics?

n.b. Term “Quantum Turbulence” due to Donnelly & Swanson (1986)

Feynman, 1955
Complex structures in quantum hydrodynamics?

n.b. Term “Quantum Turbulence”
due to Donnelly & Swanson (1986)

Feynman, 1955
Superfluid Helium

\((^4\text{He}/^3\text{He})\)

Guo et al., PRL, 2010
Experimental/Physical Systems

Superfluid Helium

$^4\text{He}/^3\text{He3}$

Salort et al., 2012

Varga et al., 2015
Experimental/Physical Systems

Atomic BECs

Henn, Bagnato et al., 2009
Experimental/Physical Systems

Atomic BECs

Integration + fit -> residuals

Serafini et al, PRX, 2017
Experimental/Physical Systems

Beyond BECs

2 component systems

Rayleigh-Taylor

Kelvin-Helmholtz

Spinor condensates, Polariton-Excitons, Lee-Huang-Yang Fluids, ...

Sasaki, 2009

Takeuchi, 2018
2\textsuperscript{nd} component for visualization?

Doran et al., in prep
**Dipolar Gases**

<table>
<thead>
<tr>
<th></th>
<th>Condensed</th>
<th>Groups</th>
<th>Magnetic moment</th>
<th>ε_{dd}</th>
</tr>
</thead>
<tbody>
<tr>
<td>Alkali metals</td>
<td>1995</td>
<td>...</td>
<td>1μ_B</td>
<td>~0.01</td>
</tr>
<tr>
<td>Chromium (^{52}\text{Cr})</td>
<td>2005</td>
<td>Pfau (Stuttgart) Laburthe-Tolra (Paris)</td>
<td>6μ_B</td>
<td>0.16</td>
</tr>
<tr>
<td>Dysprosium (^{160}\text{Dy},^{162}\text{Dy},^{164}\text{Dy})</td>
<td>2011</td>
<td>Lev (Stanford) Pfau (Stuttgart) Modugno (Florence)</td>
<td>10μ_B</td>
<td>1-1.4</td>
</tr>
<tr>
<td>Erbium (^{166}\text{Er},^{168}\text{Er})</td>
<td>2012</td>
<td>Ferlaino (Innsbruck)</td>
<td>7μ_B</td>
<td>0.4-0.9</td>
</tr>
</tbody>
</table>

- Atomic interaction potential:
  \[ U(\mathbf{r} - \mathbf{r}') = g\delta(\mathbf{r} - \mathbf{r}') + \frac{C_{dd}}{4\pi} \frac{1 - 3\cos^2\theta}{|\mathbf{r} - \mathbf{r}'|^3} \]

  with \( g = \frac{4\pi\hbar^2a_s}{m} \) and \( C_{dd} = \mu_0\mu^2 \)

- Relative dipolar-to-contact interaction strength:
  \[ \varepsilon_{dd} = \frac{C_{dd}}{3g} \]

- Control of \( \varepsilon_{dd} \) by Feshbach tuning of \( g \)

See Srivatsa Badariprasad’s talk this afternoon
• Polarization driven by pinning.

\[ L \xi^3 \propto t^{-1} \]

\[ \varepsilon_{dd} = 0 \]

\[ \varepsilon_{dd} = 0.8 \]
Stagg et al., 2017

Vinen

Kolmogorov

Muller, 2022

Roche & Barenghi, 2008
Classical (viscous) turbulence

• In a 3D classical turbulent flow, large scale eddies break up into smaller eddies, these into smaller ones and so on... *(Richardson Cascade)*

• If there is a large inertial range between the forcing and dissipation scale (i.e. high Re) then the flow of energy through scales is characterized by a constant energy flux.

• Dimensional analysis leads to a power-law scaling for the energy spectrum,

\[
E(k) = C \epsilon^{2/3} k^{-5/3}
\]

\[
\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = -\frac{1}{\rho} \nabla p + \nu \nabla^2 \mathbf{v}
\]
Classical (viscous) turbulence

\[ S_p(r) = \langle (\delta u(r))^p \rangle, \quad \delta u(r) = u(x + r) - u(x) \]

Self similarity K41:

\[ S_p(r) \sim (\epsilon r)^{p/3} \]

\[ E(k) = C\epsilon^{2/3}k^{-5/3} \]

Maurer & Tabeling, EPL, 1998
Flatness factor:

$$\frac{(\delta_x u)^4}{[\frac{\partial^n u}{\partial x^n}]^2}$$

K41 implies a constant, independent of Reynolds number...

...problem

Interestingly it was probably Batchelor who introduced K41 to the west in his 1947 paper:

“Like a prospector going through a load of crushed rock, I suddenly came across rushed rock, I suddenly came across two articles about four pages in length, whose quality was immediately clear.”
Measurements describing the probability distribution of $\frac{\partial u}{\partial x}$, $\frac{\partial^2 u}{\partial x^2}$ and $\frac{\partial^3 u}{\partial x^3}$ are also described. These, and oscillograms of the velocity derivatives, show that the energy associated with large wave-numbers is very unevenly distributed in space. There appear to be isolated regions in which the large wave-numbers are ‘activated’, separated by regions of comparative quiescence. This spatial inhomogeneity becomes more marked with increase in the order of the velocity derivative, i.e. with increase in the wave-number. It is suggested that the spatial inhomogeneity is produced early in the history of the turbulence by an intrinsic instability, in the way that a vortex sheet quickly rolls up into a number of strong discrete vortices. Thereafter the inhomogeneity is maintained by the action of the energy transfer.
Coherent structures

Siggia, JFM, 1981, $32^3$

She et al., Nature, 1990, $96^3$

Ishihara et al., $4096^3$
Aside: Book plug (zero € incentive)
Importance?

Frisch, 1995

Roussel, Schneider & Farge, 2005
Figure 2. Visualization of the vorticity amplitude of $\omega$ (a) and $\mathcal{D}_\ell^\ell$ (b) for $\ell = 8\eta$ on a plane containing the centre perpendicular to the cylinder’s axis from the DNS of table 1.
Vinen tangle – unstructured quantum turbulence

$\omega \approx 0$

AWB, 2011
Coherent structures in QT?

• Do coherent structures exist in quantum turbulence?

• What are these structures, bundles? How do they form and evolve?

• Would allow a mechanism for vortex stretching, i.e. stretch the bundle.

\[
\frac{D\omega}{Dt} = (\omega \cdot \nabla)v + \nu \nabla^2 \omega
\]
Modelling approach

3 distinct scales/numerical approaches

Gross-Pitaevskii

Point Vortex/VFM

Course-Grained HVBK

Barenghi et al. (2014)
Vortex filament method

Biot-Savart Integral

$$\frac{ds}{dt} = -\frac{\Gamma}{4\pi} \oint_L \frac{(s - r)}{|s - r|^3} \times dr$$

Model reconnections algorithmically ‘cut and paste’
Local induction approximation

\[ \frac{ds}{dt} = -\frac{\Gamma}{4\pi} \oint_{\mathcal{L}} \frac{(s - r)}{|s - r|^3} \times dr \]

arclength

\[ \xi, \quad s = r(\xi = 0), \quad r(\xi) = s + \xi s' + \xi^2 s''/2 + \ldots \]

\[ \frac{ds}{dt} \approx \frac{\Gamma}{4\pi} \oint_{\mathcal{L}} \frac{(\xi s' + \frac{1}{2} \xi^2 s'' + \ldots) \times (s' + \xi s'' + \ldots)}{|\xi s' + \frac{1}{2} \xi^2 s'' + \ldots|^3} \, d\xi \]

\[ = \frac{\Gamma}{8\pi} s' \times s'' \int_{\mathcal{L}} |\xi|^{-1}(1 + O(\xi)) \, d\xi \approx \beta s' \times s'' \]
Local induction approximation

Biot-Savart

GPE

LIA

Kursa, 2011
Problems with LIA

\[
\frac{ds}{dt} \beta s^0 \times s''
\]

\[
\frac{ds}{dt} = -\frac{\Gamma}{4\pi} \int \frac{(s - r)}{|s - r|^3} \times dr
\]

Schwarz (1988) had to introduce an artificial mixing process in his original paper in counterflow turbulence.

Adachi et al. (2010)
Desingularisation

\[ \frac{ds}{dt} = -\frac{\Gamma}{4\pi} \oint_{\mathcal{L}} \frac{(s - r)}{|s - r|^3} \times dr \]

Singular as \( r \to s \)

\[ \frac{ds_j}{dt} = v_s^{loc}(s_j) + v_s^{non}(s_j) = v_{ring}^{non} - \frac{\Gamma}{4\pi} \oint_{\mathcal{L}'} \frac{(s_j - r)}{|s_j - r|^3} \times dr \]

Rankine core assumption

\[ \frac{ds_j}{dt} = v_s^{loc}(s_j) + v_s^{non}(s_j) = \frac{\Gamma}{4\pi} \ln \left( \frac{2\sqrt{l_jl_{j+1}}}{e^{1/4}a_0} \right) s_j' \times s_j'' - \frac{\Gamma}{4\pi} \oint_{\mathcal{L}'} \frac{(s_j - r)}{|s_j - r|^3} \times dr \]
Mutual friction (see Luca’s talk)

\[ \frac{ds}{dt} = v_{s}^{\text{tot}} + \alpha s' \times (v_{n}^{\text{ext}} - v_{s}^{\text{tot}}) - \alpha' s' \times [s' \times (v_{n}^{\text{ext}} - v_{s}^{\text{tot}})] \]

Normal viscous fluid coupled to inviscid superfluid via mutual friction.

Superfluid component extracts energy from normal fluid component via Donelly-Glaberson instability, amplification of Kelvin waves.

Kelvin wave grows with amplitude \( A(t) = A(0)e^{\sigma t} \)

\[ \sigma(k) = \alpha(kV - v'k^2) \]

\( v_{n}^{\text{ext}}(s, t) = (c, 0, 0) \)
Fluctuations of vortex line density scale as $f^{-5/3}$.
If we interpret L as a measure of the rms superfluid vorticity.
Contradiction of the classical scaling of vorticity expected from K41.
Roche & Barenghi (EPL, 2008) - vortex line density field is decomposed into a polarised component, and a random component.
Random component advected as a passive scalar explaining $-5/3$ scaling.

Roche et al., EPL, 2007
Detecting structures

\[ \omega(r, t) = \kappa \sum_{i=1}^{N} \frac{s'_i}{(2\pi \sigma^2)^{3/2}} \exp(-|s_i - r|^2 / 2\sigma^2) \Delta \xi \]

\[ \sigma = \ell \]
Quantum turbulence at finite temp.

Drive turbulence in superfluid component to a steady state with imposed normal ‘fluid turbulence’.

\[ \mathbf{v}_{n}^{ext}(s, t) = \sum_{m=1}^{m=M} (A_m \times k_m \cos \phi_m + B_m \times k_m \sin \phi_m) \]

Identify regions of high course-grained vorticity

\[ \omega(r, t) = \kappa \sum_{i=1}^{N} \frac{s_i}{(2\pi \sigma^2)^{3/2}} \exp\left(-|s_i - r|^2/2\sigma^2\right) \]

AWB, Laurie & Barenghi, 2012
Decomposition of a tangle

AWB, Laurie & Barenghi, 2012
Numerical results

Left, frequency spectra (red polarised; black total), right energy spectrum, upper random component, lower polarised component.
Comparable results using GPE
Presence of coherent structures inferred from intermittent pressure drops, HVBK:

$$\nabla^2 P = \frac{\rho_s}{2} (\omega_s^2 - \sigma_s^2) + \frac{\rho_n}{2} (\omega_n^2 - \sigma_n^2)$$
\( \omega_s \quad \quad \quad P \)

![Diagram of isosurfaces and numerical data graph](image)

\[ P \sim P_0 - \frac{\rho_s N^2 \Gamma^2}{8\pi^2 r^2}, \]

**FIG. 5.** Isosurfaces of the coarse-grained standardized vorticity magnitude (left) and negative pressure (right) fields of the static quasiclassical tangle depicted in Fig. 3 using \( l_f = 2\ell \). The isosurfaces are taken at \( \omega/\omega_{\text{vort}} > 2.5 \) and \( P/\sigma_{\text{press}} < -1.5 \), respectively.

Baggaley & Laurie, 2020
Vortex reconnections

Now we have motivated the presence of vortex bundles what about their interaction? Define $\delta$, minimum separation
Vortex reconnections

Now we have motivated the presence of vortex bundles what about their interaction? Define $\delta$, minimum separation

$$\delta(t) = (C_1 \kappa)^{1/2} t^{1/2} \quad \text{interaction regime,}$$

$$\delta(t) = C_2 \left( \frac{\kappa}{\ell} \right) t \quad \text{driven regime,}$$

$$\delta(t) = C_3 \left( \frac{\nabla \rho}{\rho} \right) t \quad \text{driven regime,}$$

Galantucci et al., 2020

Yao & Hussain 2020
Different Scaling Exponents

Yao & Hussain, 2020
Approach and separation of bundles of quantized vorticity

George S. E. Grimes* and Andrew W. Baggaley†

Joint Quantum Centre Durham–Newcastle, School of Mathematics and Statistics, University of Newcastle, Newcastle upon Tyne NE1 7RU, United Kingdom

FIG. 1. Snapshots from a numerical simulation of the reconnection of vortex bundles. Here the initial bundle separation is $D = 0.1\text{cm}$, the bundle radius is $A = 0.03\text{cm}$ and the number of vortices in each bundle is $N = 8$. 

Grimes, AWB, 2022
Dimensionless Variables

\[ \delta(t = 0) = D \]
\[ \tau = \frac{D}{v_\theta} = \frac{2\pi D A}{N \Gamma} \]
\[ \delta^* = \frac{\delta}{D} \]
\[ t^* = \frac{t}{\tau} \]
Varying cross-sectional area, $A$

Dimensionless Variables

\[ \delta(t = 0) = D \]

\[ \tau = \frac{D}{v_{\theta}} = \frac{2\pi DA}{N\Gamma} \]

\[ \delta^* = \delta/D \]

\[ t^* = t/\tau \]
Bridges

In viscous reconnections we see a viscous bridge form, our simulations show an analogous process can occur in quantum fluids.
Future Avenues

Figure 12
(a) Two vibrating-wire resonators side by side. If one (the generator wire) is oscillating fast enough to produce vorticity, the resultant cloud covers the neighboring (or detector) wire and shields it from the damping effect of the ambient thermal excitations. (b) The effect of vorticity on the damping of the detector wire for two generator wire velocities. The velocity is indicated in each panel as a fraction of the Landau velocity, $v_L$, described in Section 3.2. The generator is switched on during the interval indicated in blue. In the upper data, the generator wire is not producing vortices but is pair breaking, and the detector wire simply sees the larger damping of the greater excitation density. In the lower data, the generator wire is producing vortices (and even more broken pairs), but despite the even higher excitation density the vorticity smothers the damping, which falls (in this case by $\sim 10\%$).

Vortex generation above the critical velocity, a compact region of turbulence can be created. The turbulent flow fields, when illuminated by an excitation beam, reflect a fraction of the excitations in the beam back into the radiator container, as shown in the schematic in Figure 13a. The results are shown in Figure 13b. When the turbulence is illuminated with an excitation beam the fraction of the beam reflected can be measured. This is shown in Figure 13b (upper). No Andreev reflection is seen until the generator wire exceeds the critical velocity for vortex creation. In other words, with no vortices there is no reflection. Furthermore, this is not a small effect. The fraction reflected extends beyond 25%. Because the reflection cross-section for a length of vortex line is known ($\sim 20 \mu m$), as is the rough size of the vortex tangle (a few millimeters), the reflection fractions can be converted to vortex-line density as shown in Figure 13b (lower).

A typical value for the density in the middle of the figure ($2 \times 10^7 m^{-2}$) corresponds to a vortex separation of around 0.2 mm. Densities 20 times smaller than this can be readily resolved, corresponding to a vortex separation of about 1 mm. That is unbelievably dilute, given that the tangle is only a few millimeters across. This provides a clue to how potentially sensitive these methods can be.

A similar experiment has been made in one of the Helsinki rotating cryostats (19, 20) with a regular array of vortices rather than with a vortex tangle. The experimental cell is shown in Figure 14a. The upper chamber acts as the bolometer, and excitations emitted through the hole to the lower chamber are Andreev reflected by the array of vortices stabilized by rotation. (Of
Knots (& Braids) in Nature…

Bird Flocks?

DNA

Material Science

Solar Flares

\[ \eta \]

\[ \epsilon \]

\[ \dot{\eta} \]

\[ \epsilon_1 = +1 \]

\[ \epsilon_2 = -1 \]

\[ \epsilon_3 = +1 \]
...and in fluids

Kleckner & Irvin, 2013
...and in quantum fluids

Proment et al., 2012
Confined normal fluid

\[
\frac{ds}{dt} = v_{s}^{\text{tot}} + \alpha s' \times (v_{n}^{\text{ext}} - v_{s}^{\text{tot}}) - \alpha' s' \times [s' \times (v_{n}^{\text{ext}} - v_{s}^{\text{tot}})]
\]

\[
v_{n}(r, \theta, \phi) = \sum_{l,m} t_{l}^{m} + s_{l}^{m},
\]

where,

\[
t_{l}^{m} = \nabla \times \hat{r}_{l}^{m} Y_{l}^{m}(\theta, \phi), \quad s_{l}^{m} = \nabla \times \nabla \times \hat{r}_{l}^{m} Y_{l}^{m}(\theta, \phi), \quad -l \leq m \leq l.
\]

Here we consider the following Dudley & James flow:

\[
v_{n} = t_{2}^{0} + \varepsilon s_{2}^{0},
\]
Knot Polynomials

- Break each snapshot of the turbulent tangle into a set of distinct loops $\mathcal{L}_j$
- Determine each loops Alexander polynomial
  \[ \Delta_j = a_0 + a_1 \tau + \ldots + a_{\nu_j} \tau^{\nu_j} \]
- The degree of the polynomial quantifies the complexity of the knot.
- Computed via numerical algorithm

\[ \Delta(\tau) = 1 - \tau + \tau^2 \]

\[ \nu_j = 82 \]
Knot Polynomials

- Break each snapshot of the turbulent tangle into a set of distinct loops \( \mathcal{L}_j \)
- Determine each loops Alexander polynomial
  \[ \Delta_j = a_0 + a_1 \tau + \ldots + a_{\nu_j} \tau^{\nu_j} \]
- The degree of the polynomial quantifies the complexity of the knot.
- Computed via numerical algorithm

\[ \Delta(\tau) = 1 - \tau + \tau^2 \]
\[ \Delta(\tau) = 1 \Rightarrow \nu = 0 \]
Two simulations – different drives

$\Lambda$ – total vortex line length

Figure 2. Time evolution of the vortex line length $\Lambda$ (cm) in the two numerical simulations. The lower/upper (blue/red) lines correspond to normal fluid drives of $v_f = 4.75$ cm/s and $v_f = 5.25$ cm/s respectively.

Figure 3. Snapshots of the vortex configuration for the lower (left) and higher (right) normal drives $v_f = 4.75$ cm/s and $v_f = 5.25$ cm/s respectively at $t = 20s$ in the saturated steady-state regime. Different colour sar e used to identify distinct vortex lines in the two snapshots. The cubes around the vortex angles are for visualization only (the simulations are performed in an infinite domain).

Figure 6. Time series of the largest degree of Alexander polynomial in the vortex configuration. The upper line is for $v_f = 4.75$ cm/s, and the lower line for $v_f = 5.25$ cm/s.
Knot Probability

For headphones the probability of a knot is close to 1, what about our vortices?

\[ P_k = \frac{1}{1 + (\Lambda_i/\Lambda_0)^\gamma} \]

Arsuaga et al., 2002

Raymer & Smith, 2007
IG Nobel 2018
Figure 8. Probability mass functions (PMFs) of the degree of the Alexander polynomial $\nu$ plotted on a log-log scale (main graph) and linear-linear scale (inset). The logarithmic scale suggests $PMF(\nu) \sim \nu^{-3/2}$, as shown by the black dashed line.
The End